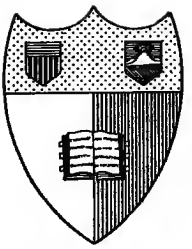


Cornell University Library

Ithaca, New York

BOUGHT WITH THE INCOME OF THE

SAGE ENDOWMENT FUND

THE GIFT OF

HENRY W. SAGE

1891

Cornell University Library
arV18750
The transition spiral and its introducti
3 1924 031 215 142
olin,anx

The original of this book is in
the Cornell University Library.

There are no known copyright restrictions in
the United States on the use of the text.

http://www.archive.org/details/cu31924031215142

THE TRANSITION SPIRAL

THE TRANSITION SPIRAL

AND

ITS INTRODUCTION TO RAILWAY CURVES

WITH

FIELD EXERCISES IN CONSTRUCTION AND ALIGNMENT

BY

ARTHUR LOVAT HIGGINS

M.Sc., A.R.C.S., A.M.Inst.C.E.

Assistant-Lecturer in Engineering in the Victoria University of Manchester, and formerly Assistant to the Professor of Civil Engineering in the Queen's University of Belfast, Author of "The Field Manual"

NEW YORK

D. VAN NOSTRAND COMPANY

EIGHT WARREN STREET

Printed in Great Britain

PREFACE

THE aim of this handbook is to afford concise and complete demonstration of the introduction of transition curves, or spirals, in the various alignment problems that accrue in connection with the construction or improvement of railway curves.

Now the extent to which such problems can be satisfactorily essayed depends upon the mathematical development of the form of transition adopted, and the facility with which this development admits utilisation in the field—the effective compromise of mathematical truth with practical precision.

Hence, besides fulfilling its objective function, the ideal transition curve must be easy and flexible in construction; a curve with simple, definite relations to main curves and straights; and, above all, one which can be introduced without the aid of special tables, such as hamper the construction of American tabular spirals.

The curve most readily amenable to these requirements is to be found in the clothoid $\lambda = m\sqrt{\phi}$, the spiral advanced and investigated in this connection by Mr. James Glover, M.A., A.M.I.C.E., in a paper read to the Institution of Civil Engineers.*

This spiral† yields an approximation that possesses essentially simple properties, and, without introducing appreciable working error, lends itself to construction

* *Transition Curves for Railways*, Proc. Inst. Civ. Engs., vol. cxl. (June, 1900).

† Ex "*Sino Loria.*" First referred to by Jas. Bernouilli (1654-1705); rediscovered by Cornu (1874), and called *Cornu's spiral* in optics; and fully discussed by Cesaro (1890), who named it the "clothoid."

in so many ways that it may justly be styled the *transition spiral.*

1. It is as easily applied to the degree system as to the radius system, and, adapting itself to the curve, is superior to tabular polychord spirals which form the bulk of the multiform compound curves largely employed in America.

2. It may be set out by linear means solely with the theodolite, or, if occasion arise, with two theodolites, no special tables being required in its various modes of construction.

3. It is virtually interchangeable with the cubic parabola, and may thus supply dimensions not regularly associated with the latter form of transition curve.

4. Its relations with tangent and circle, or circle and circle, are simple and definite, and thus facilitate its introduction to all cases of alignment.

The following pages contain an essay on the length of transition curves, a complete demonstration of the various modes of utilising the clothoid as a transition curve, a systematic investigation of the introduction of the spiral to the various problems of alignment, and a selection of field exercises, these last being an effort to present the real matter, the actual operations, in practical form.

The author desires to express his indebtedness to the various sources of information named in the text, and to Messrs. W. F. McMurray, W. E. English, G. Trotter, and J. D. Gordon, of Queen's University, Belfast, for their assistance in the preparation of the tables and drawings.

ARTHUR LOVAT HIGGINS.

QUEEN'S UNIVERSITY,
BELFAST.
March 1, 1920.

CONTENTS

INTRODUCTION

PAGE

Superelevation and cant—Superelevation and speed—Tangent and curve—Transition—Spiral elements—Length of transition: Uniform approach; time approach; rules—Tables: Data of time approaches in the radius and degree systems - - - - 1

PART I

DEVELOPMENT

Intrinsic equation of clothoid—Derivation of cartesians and deflections—Chord—deflection approximation and its three simple properties—Investigation of modes of construction—Construction as polychords; "spiralling through" and "spiralling in"—Normal deflections and the occupation of spiral points as stations—Interpolating stations—Construction by offsets—Central transition of compound curves—Conclusion—Tables of appendix explained—Spiral notation - - - - 13

PART II

ALIGNMENT

Classification of problems—The general case of alignment—Transitions replacing the point of compound curvature—The four general problems applied to simple, compound, and reverse curves—Equation of equal routes - - 35

PART III

FIELD EXERCISES

PAGE

[*Problem*] (1) Spiralling through to a simple curve; radius unaltered—(2) Spiralling in to a simple curve; interpolating stations—(3) Offsetting a transition to a simple curve —(4) Replacing a compound by branches of the same degree, but suitably spiralled—(5) Replacing a compound by branches of the same radii, but with central and end transitions—(6) Superseding a simple curve by one with spirals, the greater portion of the original curve remaining undisturbed—(7) Replacing a compound by one with spirals, the point of compound curvature remaining undisturbed—(8) Replacing a reverse curve by two simple curves with end transitions and spiral reverse—(9) Replacing a simple curve by a spiralled curve, the latter approximating the original alignment—(10) Replacing a compound by one with spirals that affords the same length of route - - - - - - - - 51

APPENDIX

Table I., Spiral elements and data—*Radius system* - 100
Table II., Spiral elements and data—*Degree system* - 104

INDEX - 109

THE TRANSITION SPIRAL

INTRODUCTION

Superelevation and Cant.—Cant is the transverse slope given to the surface of the rails on a curve in order to counteract, in particular,* the effect of the centrifugal force of the moving train.

Commonly, however, the term is used synonymously with "superelevation," which is the amount by which the outer rail is raised accordingly.

Thus, if v is the velocity of a train in feet per second, when moving round a curve r feet in radius, the resulting centrifugal force is to the weight of the train as $\frac{v^2}{32 \cdot 2r}$: 1; and this is the ratio that the superelevation must bear to the horizontal distance between the centres of the rails, in order that the resultant force shall act perpendicularly to the surface of the rails and so avoid pressure between the wheel flanges and the outer rail. Therefore, if V be the velocity in miles per hour, the superelevation e in feet required on a curve of R chains radius is expressed very nearly by the following relation,

* The modern bogey obviates the additional cant once necessary on account of parallelism of axles and slip of wheels, leaving centrifugal force alone to be considered.

G the distance in feet between the rail centres differing inappreciably from its horizontal equivalent—

$$e=G\frac{V^2}{990R}; \tag{1}$$

or, in the degree system,

$$e=G\frac{V^2D}{85950}, \tag{2}$$

which for the standard gauge of 4′ 8½″ and ordinary railheads reduce respectively to

$$e=\frac{V^2}{200R} \tag{3}$$

and

$$e=\frac{V^2D}{17500}. \tag{4}$$

In practice the superelevation is commonly specified in inches.

Superelevation and Speed.—Since it is impossible to adjust the cant for all speeds, it is best to adapt it to nearly the highest speed occurrent on the line. In practice, however, the superelevation is limited by (*a*) the comfort of the passengers, (*b*) the case of stoppages on the curve, (*c*) the effect of wind pressure, and (*d*) the contingency of the outer wheel flange mounting the rail.*

Six inches appears to be generally favoured as the maximum amount of superelevation permissible; but this limit is often carried to as much as 8 inches. The maximum permissible superelevation may be

* The limitations of superelevation are thoroughly investigated in an excellent paper by Mr. J. W. Spiller, A.M.I.C.E., in the Proceedings of the Institution of Civil Engineers: *High Speed on Railway Curves*, vol. clxxvi. (1909).

regarded as fixing the maximum allowable speeds on curves in accordance with the equation $V_m = \sqrt{200eR}$; but not rigidly, as a somewhat greater velocity, about $\frac{11}{10}$ths, is allowed in practice, causing a lateral pressure on the outer rail, and displacing the centre of pressure slightly laterally.*

In fact, *eleven* times the *square root* of the *radius* in *chains* appears to be the *maximum speed* in *miles per hour* that is regularly attained on curves. That is:

$$V_m = 11\sqrt{R}, \qquad (5)$$

which reduces to $102{\cdot}5 \div \sqrt{D}$ in the *degree* system, or, more conveniently,

$$V_m = \frac{100}{\sqrt{D}}. \qquad (6)$$

Tangent and Curve.—It is evident that the tangent point of a circular curve requires two conditions that cannot be realised simultaneously:

(i.) The rails should be level transversely, the tangent point being on a straight line;

(ii.) The rails should be canted in accordance with the speed on the curve.

It is thus essential that the full superelevation of the outer rail should be attained by the time the train enters the circular portion. Before the introduction of transition curves, it was customary to effect this—in opposition to (i.)—by commencing the elevation of the outer rail on the tangent itself, with the result that a sharp jerk occurred when the train suddenly entered the curve from the tangent. Besides, the

* N.B.—$11\sqrt{R}$ corresponds to a superelavation of 0·6 foot for centrifugal force.

widening of gauge for curves was gradually applied on the straight, so that the requisite width was attained at the tangent point.

Transition.—The objects of a transition curve are therefore:

1. To afford a gradual increase of curvature from zero at the tangent to the specified radius or degree of the circular portion, or main curve; and

2. To afford the elevation of the outer rail in accordance with such curvature, so that the full superelevation is attained simultaneously with the curvature of the circular portion.

The effect of introducing transition curves, or spirals, to a circular curve of given radius is to displace that curve, or to change its radius, or to produce both displacement and change.

Fig. 1 illustrates the fundamental case, a simple curve being replaced by one of the same radius provided with suitable spirals. Here the apex of the curve is shifted from v to V, moving the tangent point from a on the *main tangent* to a point A on the *shift tangent*, the central angle Δ being reduced by 2Φ, where Φ is the total angle of the spiral.

The perpendicular distance between the main tangent and the shift tangent is styled the *shift*. This quantity s, with the radius R of the main curve, constitute what may be termed the general specification of a transition curve. It is the characteristic element of such curves. Whether the main curves be simple, or compound, or reversed, and whatever be the form of transition employed, the shift bisects the transition curve, and the transition curve bisects the shift.

Thus, in Fig. 1 the transition curve PQ passes through the point R, bisecting AS, while the length PRQ is itself bisected by AS.

Spiral Elements.—The elements of a spiral are those dimensions which determine the limits and ruling points

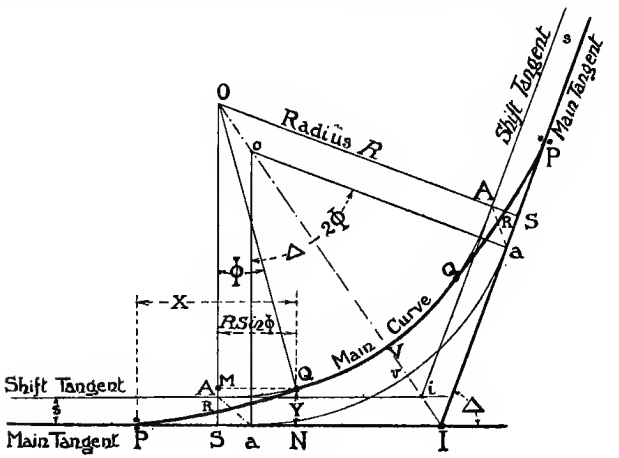


FIG. 1.

of spiral approaches without regard to the data by which the spiral itself will be constructed. These are in general the following: (Fig. 1)

1. *The Length* L.

2. *The Shift* $s = Y - R \text{ vers } \Phi$.

3. *The Shift Increment,* $aS = s \tan \frac{1}{2}\Delta$, Δ being the angle of intersection.

4. *The Spiral Extension,* $SP = X - R \sin \Phi$, Φ being the total angle of the spiral.

5. *The Tangent Excess,* $aP = X - R \sin \Phi + s \tan \frac{1}{2}\Delta$, being the sum of the increment and extension.

6. *The Total Tangent Length,* $T_s=(R+s)\tan\frac{1}{2}\Delta+X-R\sin\Phi$, which is the sum of the tangent excess and the normal length for a circular curve of radius R.

7. X, *the terminal x abscissa.*

8. Y, *the terminal y ordinate.*

9. *The Long Chord* PQ and its *Total Deflection,* the terminal deflection angle Ω. $PQ=X\sec\Omega$.

It will be seen that the tangent distance IS is equal to $(R+s)\tan\frac{1}{2}\Delta$, and, this corresponding to a curve of radius $(R+s)$, affords another aspect of regarding the case in question. (Fig. 1)

Length of Transition Curves.—Considerable divergence of opinion exists as to what constitutes the proper length of a transition curve, numerous rules being advanced, and these not infrequently being conflicting. It is, therefore, advisable at this juncture to reduce the subject to a tangible basis, bearing in mind that local conditions may render an otherwise suitable length impracticable.

Existing rules may be grouped with reserve into what may be styled (1) *Uniform Approaches,* and (2) *Time Approaches.*

1. Uniform Approaches.—Some rules assume a uniform rate of elevation gradient, running up the required superelevation in the length of the transition curve—1 in 300; 1 in 500; 1 in 1,000, as the case may be. Here it is assumed that the superelevation is given in accordance with the allowable speed, e being equal to $m\frac{V^2}{R}$ or nV^2D, where m and n are respectively equal to $\frac{G}{990}$ and $\frac{G}{85950}$. (See p. 2.)

Hence if L and l are lengths of transition corresponding to radii R and r and speeds V and v, then for the same gauge, $\frac{L}{E}$ and $\frac{l}{e}$ equal a constant k, E and e being the cants accordingly.

That is—

$$\frac{LR}{mV^2}=\frac{lr}{mv^2};$$

and

$$\frac{L}{l}=\frac{V^2r}{v^2R};$$

or

$$\frac{L}{l}=\frac{V^2D}{v^2d},$$

D and d being the degrees taken as $5730 \div$ radius.

Thus, for the same radius the lengths of transition would vary as the squares of the speeds employed.

2. Time Approaches.—The rational rule for varying speeds is that the same amount of superelevation should be attained in the same time.

That is—

$\frac{mV^2}{R}$ is the superelevation attained in time $\frac{L}{V}$,

and $\frac{mv^2}{r}$ is the superelevation attained in time $\frac{l}{v}$.

Hence the superelevation attained in unit time is $\frac{mV^3}{LR}$ and $\frac{mv^3}{lr}$; and, this being the same in each case,

$$\left.\begin{aligned} &\frac{L}{l}=\frac{V^3r}{v^3R} \quad \text{and} \quad L=a\frac{V^3}{R} \\ \text{or} \quad &\frac{L}{l}=\frac{V^3D}{v^3d} \quad \text{and} \quad L=a'V^3D \end{aligned}\right\} \qquad (7)$$

Thus, for the same radii the lengths of transition would vary as the cubes of the speeds employed.

Hence it follows if a maximum superelevation, 6″, say, be given consistently, the maximum speeds $11\sqrt{R}$ or $\frac{100}{\sqrt{D}}$ being permissible,

$$\left.\begin{array}{lll} & \frac{L}{l}=\frac{\sqrt{R}}{\sqrt{r}} \quad \text{and} \quad L=a\sqrt{R} \\ \text{or} & \frac{L}{l}=\frac{\sqrt{d}}{\sqrt{D}} \quad \text{and} \quad L=\frac{a'}{\sqrt{D}} \end{array}\right\} \qquad (8)$$

And, conversely, under these conditions the shift s and the superelevation E will be constant.

Shortt's Rule.—Mr. W. H. Shortt, A.M.I.C.E., considers the length essential for a transition curve from an entirely new standpoint, and advances that this should be such as will eliminate the sense of discomfort inherent in passing from straight to circle, the rate of gain of acceleration serving as a measure of the amount of discomfort experienced in passing the tangent point.*

Thus, if a car c feet in length moves at a velocity of v feet per second on a circle of r feet radius, the radial acceleration is $\frac{v^2}{r}$, this being gained in $\frac{c}{v}$ seconds, the time the car takes to pass the tangent point; and, assuming the rate of gain of acceleration to be uniform during the time the car is passing the tangent point, its value is $\frac{v^2}{r}$ divided by $\frac{c}{v}$; that is $\frac{v^3}{cr}$, which also may be regarded as expressing the degree of sensational discomfort.

Therefore, when there is a transition curve of length

* *A Practical Method for the Improvement of Existing Railway Curves*, Proc. Inst. Civ. Engs., vol. clxxvi. (1909).

l feet, it follows that the acceleration will have to be gained at the rate of $\frac{v^3}{lr}$ feet per second^2 in a second, assuming it to be gained uniformly. Now, if $\frac{1}{a}$ be the maximum rate of change of acceleration that will pass unnoticed, $\frac{1}{a}$ should then equal $\frac{v^3}{lr}$; and, therefore, the length l of the transition curve should equal $\frac{av^3}{r}$.

Thus, for all transition curves to have an equal effect, their lengths must be proportional to the cube of the velocity and inversely as the radius of the circular curve.

Hence, Mr. Shortt's rule is rational in the sense of time approach, while his observations from the standpoint assumed afford a value for the constant a in expressions (7) and (8) of pp. 7 and 8, which may be written accordingly $l=\frac{av^3}{r}$ and $a\sqrt{r}$ respectively.

However, in Mr. Shortt's experience, a rate of change of acceleration of 1 foot per second^2 in a second is the maximum that will pass unnoticed; and, in consequence, a may be taken as *unity*, the length of the transition curve becoming $\frac{v^3}{r}$.

Or, changing the units, V being the speed in miles per hour, R the radius in chains, and L the length of transition in chains,

$$L=\frac{V^3}{1381R}. \qquad (9)$$

Since, however, the speed may vary, the fastest trains must be considered when calculating the length

of the transition curve. But $11\sqrt{R}$ represents the maximum speeds in miles per hour that are regularly attained on curves in practice, and in consequence V may be eliminated from Eq. 9, leaving L equal to $\cdot 96\sqrt{R}$; that is, for practical purposes—

$$L=\sqrt{R}, \tag{10}$$

which expresses the length in *feet* as *six times* the *maximum speed*, the time approach length advanced by Mr. J. R. Stephens, an American authority.*

Accordingly the corresponding dimension in the degree system is conveniently

$$L'=\frac{600}{\sqrt{D}} \text{ feet}, \tag{11}$$

Now the relation $V=11\cdot 13\sqrt{R}$ obtains only up to speeds of about 82 miles per hour, and Eq. 10 therefore applies only to curves under 54·3 chains radius, lengths exceeding the square root of this limit being wasteful, as also are those calculated accordingly for speeds considerably under $11\sqrt{R}$.

However, the length of transition suitable for any other velocity, V miles per hour, can be obtained from Eq. 9; and in the case of curves over 54·3 chains radius it is merely necessary to substitute V=82, such speed being possible, reducing the expression to

$$\left.\begin{aligned} L&=\frac{400}{R} \text{ chains}, \\ \text{or} \quad L'&=300D \text{ feet } \textit{nearly}. \end{aligned}\right\} \tag{12}$$

Mr. Shortt measures the efficiency of a transition

* *The Six Chord Spiral*, by J. R. Stephens, C.E.

curve by the reciprocal of the rate of change of acceleration, which, written as $\frac{1381LR}{V^3}$, gives values greater or less than unity according as lengths L longer or shorter than $\sqrt{R}$ are employed. Although somewhat anomalous, such usage of the term "efficiency" is defensible, being sufficiently incongruous to emphasise that a transition is uselessly or unnecessarily long.

Tables.—The following tables exhibit in the radius and degree systems the relations between the various data of time approaches, the lengths of transition being taken respectively as $\sqrt{R}$ chains and $6V_m$ feet, in accordance with maximum speeds V_m of $11{\cdot}13\sqrt{R}$ and $100 \div \sqrt{D}$ miles per hour. Under these conditions the shift s is constant, being 33 inches and $31\frac{7}{16}$ inches respectively, as in the relation $s = \frac{L^2}{24R}$.

TABLE I.

RADIUS SYSTEM.

Radius R Chains.	*Max. Speed V M.P.H.*	*Length of Transition L Chains.*	*Superelevation E Inches.*	*Normal Shift s.*	*Gradient of Adjustment.*
20	49·8	4·472	6 inches	2 feet 9 inches, or $\frac{1}{24}$ chain	1 in 580
25	55·6	5·000			1 in 660
30	61·0	5·477			1 in 723
35	65·8	5·916			1 in 781
40	70·4	6·325			1 in 835
45	74·7	6·708			1 in 885
50	78·7	7·071			1 in 933
55	82·5	7·416			1 in 979
60	*86·2*	*7·746*			*1 in 1022*
80	*100·0*	*8·944*			*1 in 1181*

TABLE II.

DEGREE SYSTEM.

Degree D.	*Max. Speed V M.P.H.*	*Length of Spiral L Feet.*	*Superelevation E Inches.*	*Normal Shift s.*	*Gradient of Adjustment.*
1	*100·0*	*600·0*	6 inches	31·416 inches, or 2 feet $7\frac{7}{16}$ inches	*1 in 1200*
2	70·7	424·2			1 in 848
3	57·7	346·2			1 in 692
4	50·0	300·0			1 in 600
5	44·7	268·2			1 in 536
6	40·8	244·8			1 in 490
7	37·7	226·5			1 in 452
8	35·4	212·4			1 in 425
9	33·3	199·8			1 in 400
10	31·6	189·6			1 in 379

N.B.—It is evident that the length of transition in feet in either table is equal to the maximum speed multiplied by the full superelevation in inches. Hence, if a curve is elevated by an amount e less than 6 inches for a speed V less than V_m, the necessary length of transition can be found by multiplying the speed V by the superelevation e in inches.

PART I

DEVELOPMENT

THE present section of this work deals with the development of the clothoid as a railway transition curve.

The intrinsic equation of this spiral is—

$$\lambda = m\sqrt{\phi}\ ; \qquad \text{(i.)}$$

where λ is the distance of any point on the curve as measured along the latter from its origin, m a constant for any particular curve, and ϕ the angle between the tangent at any point and the tangent at the origin—an angle which in the present connection may be styled the spiral angle.

Now the radius of curvature ρ of any point on the spiral is

$$\frac{d\lambda}{d\phi};\ \text{and}\ \frac{d\lambda}{d\phi} = \frac{m}{2\sqrt{\phi}}; \qquad \text{(ii.)}$$

whence on substituting for ϕ,

$$\rho = \frac{m^2}{2\lambda}, \qquad \text{(iii.)}$$

which indicates that the radius of curvature varies inversely as the distance measured along the curve from the origin, being infinite when λ is zero, and r

when λ becomes l, the spiral merging into a circle of radius r. (Fig. 2)

Hence, since $\rho=r$ when $\lambda=l$,

$$m=\sqrt{2rl}, \qquad \text{(iv.)}$$

thus leading to an equation in terms of the data, the following—

$$\lambda=\sqrt{2rl\phi}. \qquad (1)$$

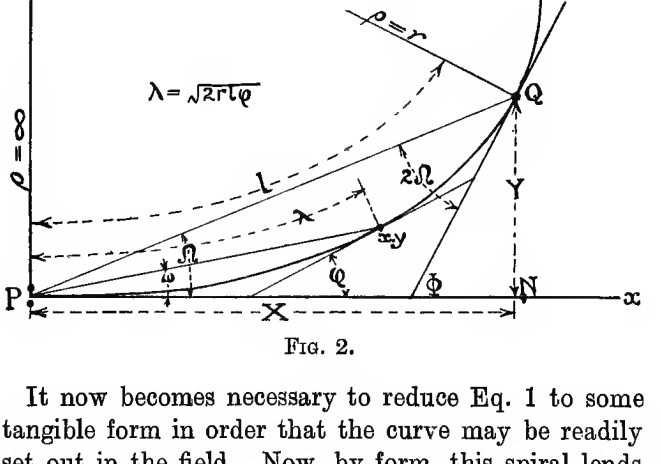


Fig. 2.

It now becomes necessary to reduce Eq. 1 to some tangible form in order that the curve may be readily set out in the field. Now, by form, this spiral lends itself the more readily to construction in a manner akin to that of setting out circular curves with the theodolite, a series of equal chords c being laid down with deflection angles ω, the latter being measured from the tangent at the origin P (Fig. 2). This necessitates the determining of the cartesians x and y, $\frac{y}{x}$ being equal to tan ω.

Now $dx=d\lambda \cos \phi=\frac{m \cos \phi d\phi}{2\sqrt{\phi}}$; hence by expanding cos ϕ and integrating,

$$x=m\sqrt{\phi}\left(1-\frac{\phi^2}{5\times 2!}+\frac{\phi^4}{9\times 4!}-\frac{\phi^6}{13\times 6!}\cdots\right). \quad \text{(v.)}$$

Similarly, $dy=d\lambda \sin \phi=\frac{m \sin \phi d\phi}{2\sqrt{\phi}}$; and by expanding sin ϕ and integrating,

$$y=m\sqrt{\phi}\left(\frac{\phi}{3}-\frac{\phi^3}{7\times 3!}+\frac{\phi^5}{11\times 5!}-\frac{\phi^7}{15\times 7!}\cdots\right). \quad \text{(vi.)}$$

Therefore,

$$\tan \omega=\frac{y}{x}=\frac{\phi}{3}+\frac{\phi^3}{105}+\frac{\phi^5}{5997}+\frac{\phi^7}{198700}+\cdots \quad \text{(vii.)}$$

Now on expanding tan ω, it can be shown that $\omega=\frac{\phi}{3}$ for all practical purposes, seldom exceeding 6°, or, roughly, a tenth of a radian.

1. *Thus the total deflection angle is one-third of the total angle of the spiral.*

Hence on substituting $\lambda=l$ and $\phi=\Phi$ in Eq. 1,

$$l=2r\Phi \quad (2)$$

and

$$\Omega=573\frac{l}{r}, \quad (3)$$

Ω being the total deflection angle expressed in minutes.

Further, Eq. 1 may be written as follows, *nc* approximating the distance λ in n chords of length c—

$$\omega=573\frac{n^2c^2}{rl} \text{ minutes;} \quad (4)$$

Or, generally, $\omega=n^2k$ minutes, (5)

where k is a constant depending upon the method by which the curve is constructed.

2. *Thus the deflections from the tangent at the origin vary as the squares of the number of chords or units of distance along the spiral.*

Also, if Y be the *terminal ordinate* in Eq. vi.—

$$Y = l\left(\frac{\phi}{3} - \phi\frac{3}{42} + \dots\right).$$

But $\Phi = \frac{l}{2r}$; hence—

$$Y = \frac{l^2}{6r}\left(1 - \frac{\Phi^2}{14} + \dots\right) = \frac{l^2}{6r} \text{ very nearly,} \qquad (6)$$

the ratio $1 - \frac{\Phi^2}{14} : 1$ being ·99922 when $\Phi = 6°$.

Further, if ϕ is small, ϕ^2 is negligible, as also are higher powers.

Then $x = m\sqrt{\phi}$ and $y = \frac{m}{3}\phi^{\frac{3}{2}}$; (viii.)

whence $y = \frac{x^3}{3m^2}$, a *cubic parabola.*

But $m^2 = 2rl$;

therefore $y = \frac{x^3}{6rl}$. (7)

Now the length of circular curve displaced by the transition curve is exactly one-half the length of the latter, these values being respectively $r\phi$ and $2r\phi$ by the spiral and circle accordingly.

Hence $QN = Y = AS + \frac{(AQ)^2}{2r}$ (Fig. 1)

$= \text{Shift } s + \frac{l^2}{8r}$ by the circle.

But $Y = \frac{l^2}{6r}$ by the spiral.

Therefore $s = \frac{l^2}{6r} - \frac{l^2}{8r} = \frac{l^2}{24r}$. (8)

In fact, the last relation applies more closely to the clothoid than to the cubic, the curve with which it is usually associated.

3. *Thus the elements of the clothoid are for the lengths of transition occurrent in practice interchangeable with those of the cubic parabola, which, not without reason, remains the engineer's conception of the ideal transition curve.*

Construction.—The methods by which this curve can be constructed in the field are substantially two—

I. By deflection angles from the main tangent, using the theodolite and chain.

II. By rectangular co-ordinates, offsetting the curve from the main tangent as though it were Froude's parabola.

I. By Deflections.—The clothoid lends itself readily to the three characteristic modes utilised in constructing American Polychord Spirals—viz., (1) *Constant chord length*, (2) *Number of chords constant*, and (3) *Constant deflection angles.*

In all these an integral number of chords is understood, and the points located are termed "spiral points," as distinct from stations, which latter are to be interpolated, often by an approximate process, with the result that spiral points alone can be occupied in the case of obstructions.

1. *Constant Chord Length.*—Using consistently as the chord c an even fraction of a chain or 100 feet, and calculating the deflection in minutes from $\frac{573c^2}{LR}$ or $\frac{10c^2D}{L}$, L and c being in chains and hundreds of feet accordingly.

In general, a chord length of about 30 feet is desirable; hence 33 feet in the radius system and $33\frac{1}{3}$ feet in the degree system. Here, however, the length of transition may be 15 feet longer or shorter than that given tentatively by known rules.

2. *Chord Number Constant.*—Using n equal chords, 10 preferably, and calculating deflections in minutes from $\frac{573}{n^2}\frac{\mathrm{L}}{\mathrm{R}}$ and $\frac{10}{n^2}\mathrm{LD}$, L being in chains and hundreds of feet respectively.

Here, as above, the deflections will run in geometrical progression, but will normally be fractional values.

3. *Deflections Constant.*—Using a common initial deflection ω, one minute preferably, and calculating the chord length c in chains or feet from $\sqrt{\frac{\omega \mathrm{RL}}{573}}$ or $\sqrt{\frac{10\omega \mathrm{L}}{\mathrm{D}}}$ respectively.

Here, with ω one minute, the deflections will run as the square of the chord number n: 1′, 4′, 9′, 16′, etc.; but the length of transition may be as much as nearly 30 feet too long or short of the value given tentatively by known rules.

Now, a transition curve is subsidiary to a main curve, and the latter is never constructed through "curve points," the points actually located being stations of the curve. Further, an integral number of chords is desirable, though such is only of accidental occurrence in main curves. With the clothoid, however, neither condition is imposed, and the spiral may therefore be run in in a manner akin to the ranging of circular curves. Thus, the first mode may be modified as follows:

4. **Spiralling Through.**—"Chaining through" with 33 feet or $33\frac{1}{3}$ feet chords consistently, calculating the deflections in minutes per chain^2 or per 100 feet^2 from $\frac{573}{LR}$ and $10\frac{D}{L}$, L being in chains or hundreds of feet respectively.

Spiralling through is the most feasible mode of constructing transition curves, the procedure being consistent with that of setting out the main curves. The fractional deflections are certainly inconvenient, but, after all, such are common to all circular curves. Besides, latitude in selecting the value of L in either $\frac{573}{LR}$ or $10\frac{D}{L}$ allows simplification in many cases, analogous modification in the circular curve being impossible.

5. **Spiralling In.**—But in regard to simplicity and accuracy the third mode commends itself, particularly in tracing spirals before the final centring. Again, an integral number of chords with inordinate length of transition is not obligatory, a subchord removing the latter objection, while, on the other hand, both subchord and inordinate length are avoided by adopting $\frac{2}{3}'$ or $\frac{1}{3}'$ as the initial deflections, construction thus with 1′ reading theodolites being as accurate as in other modes.

Spiralling Through.—In this mode the expression $\frac{573c^2}{rl}$ reduces to $\frac{573}{rl}$, c being unity; the result—

$$k=\frac{573}{LR}; \tag{9}$$

or

$$k'=\frac{10D}{L}, \tag{10}$$

being the deflection angle in minutes per chain, 66 feet or 100 feet respectively.

Normally 33 feet or 33⅓ feet chords will be employed, giving alternate points as stations in the radius system, and third points as such in the degree system.

Radius System.—Thus, if the chainage of P, the point of spiral, is 26·82, the first point on the spiral will be Sta. 27 at the end of a subchord of ·18 chain; the next point will be at ·68 chain from P; the next, Sta. 28, at 1·18 chain from P; and so on. All that is necessary, therefore, is to tabulate the distances of the spiral points for the whole transition, and to calculate the corresponding deflections by multiplying the squares of these distances by the deflection angle per chain, which is $\frac{573}{R^{\frac{3}{2}}}$, the length L being equal to $\sqrt{R}$ chains.

Hence, if the spiral in question is to approach a 36 chain main curve, the deflections will be as follows, the deflection angle in minutes per chain^2 being $\frac{573}{36\times 6}$ or 2·653:

Spiral point ..	P	1	2	3	4	5	6
Chainage	26·82	27	27·5	28	28·5	29·0	29·5
Distance	0	0·18	0·68	1·18	1·68	2·18	2·68
Deflection angle in minutes ..	0	$(0\cdot18)^2\times 2\cdot653$ 0·09	$(0\cdot68)^2\times 2\cdot653$ 1·23	$(1\cdot18)^2\times 2\cdot653$ 3·69	$(1\cdot68)^2\times 2\cdot653$ 7·49	$(2\cdot18)^2\times 2\cdot653$ 12·61	$(2\cdot68)^2\times 2\cdot653$ 19·05

The total deflection angle Ω is also equal to $\frac{573}{\sqrt{R}} = \frac{573}{6} = 95{\cdot}50$ minutes.

"*Shifting Forward.*"—When, as in the foregoing example, the theodolite is stationed at the beginning of the spiral, the deflections from the tangent thereat to any points on the spiral may be said to be the normal deflections for those points. However, it is sometimes expedient or necessary to occupy some point on the spiral other than its beginning P, and in such cases a tangent at the point occupied should be determined, the deflections being set out from this tangent.

It follows from Eq. vii. that the tangent at any point q of the spiral can be determined by backsighting on the origin P, and laying off an angle equal to twice the normal deflection from P to q, or, expressed generally, $2q^2k$, where k is the normal initial deflection per chord or chain. But no such simple rule obtains in determining the deflections from the tangent at q to other points p of the spiral, or, conversely, the angles between the points p and the tangent at q.

Spiral point ..	7	8	9	10	11	12	Q
Chainage	30	30·5	31·0	31·5	32·0	32·5	32·82
Distance	3·18	3·68	4·18	4·68	5·18	5·68	6·00
Deflection angle in minutes ..	$(3{\cdot}18)^2 \times 2{\cdot}653$ 26·83	$(3{\cdot}68)^2 \times 2{\cdot}653$ 35·93	$(4{\cdot}18)^2 \times 2{\cdot}653$ 46·35	$(4{\cdot}68)^2 \times 2{\cdot}653$ 58·10	$(5{\cdot}18)^2 \times 2{\cdot}653$ 71·18	$(5{\cdot}68)^2 \times 2{\cdot}653$ 85·59	$(6{\cdot}00)^2 \times 2{\cdot}653$ 95·51

Both are comprehended in the general case, which involves finding the differences of the ordinates y_p and y_q and those of the corresponding abscissæ x_p and x_q, the ratio of these differences being approximately the deflections from a line through q parallel to the main tangent, the tangent at q being inclined to this parallel at an angle of $3q^2k$, or ϕ_q.

Thus:

$$\frac{y_p - y_q}{x_p - x_q} = \frac{\frac{m}{3}\phi_p^{\frac{3}{2}} - \frac{m}{3}\phi_q^{\frac{3}{2}}}{m\sqrt{\phi_p} - m\sqrt{\phi_q}} = \frac{\frac{m}{3}(3p^2k)^{\frac{3}{2}} - \frac{m}{3}(3q^2k)^{\frac{3}{2}}}{m\sqrt{3p^2k} - m\sqrt{3q^2k}}$$
$$= (p^2 + pq + q^2)k.$$

Subtracting the inclination $3q^2k$ of the tangent at q, the deflections from q to p are expressed by $(p^2 - q^2)k + q(p - q)k$ or $(q^2 - p^2)k + q(q - p)k$, according as p, the point sighted, is forward or rear of q, the point occupied.

The foregoing expressions reduce respectively to $\mathrm{N}(3q + \mathrm{N})k$ and $\mathrm{N}(3q - \mathrm{N})k$ when the points p are counted 1, 2, 3, etc., or $(p - q) = \mathrm{N}$ chords forward or rear of q, which retains its numerical value in chords or in chainage from the origin P. Spiralling through will not admit the use of the multipliers tabulated on p. 24.

Degree System.—The calculations in the degree system may be made in the manner described for the radius system, the 100 feet unit merely superseding the chain. Accordingly spiral points would occur in the main at intervals of 50 feet, which is somewhat high for precise spiralling. But the reductions for a $33\frac{1}{3}$ feet chord merely require that the station distances shall be expressed in *chordages*, the initial deflection per chord being *one-ninth* that for a 100 feet length.

Thus, if P is at 72+39, stations will occur at distances

therefrom of 61′, 161′, and 261′, the chordages of which will be 1·83 for *73*, 4·83 for *74*, and 7·83 for *75*, while intermediate spiral points will intersperse these at chordages ·83 . . . 2·83, 3·83 . . . 5·83, 6·83 . . . 8·83. Hence, if the spiral in view is to approach a 4° main curve, the initial deflection per $33\frac{1}{3}$ feet chord will be $\frac{1}{9}\ 10\ \frac{L}{D}$, or $\frac{10\times4}{9\times3}$ minutes; and the deflections will run as the squares of the chordages, viz., $(\cdot83)^2\times1\cdot481$, $(1\cdot83)^2\times1\cdot481$, $(2\cdot83)^2\times1\cdot481$, $(3\cdot83)^2\times1\cdot481$, $(4\cdot83)^2\times1\cdot481$, $(5\cdot83)^2\times1\cdot481$, $(6\cdot83)^2\times1\cdot481$, $(7\cdot83)^2\times1\cdot481$, and $(8\cdot83)^2\times1\cdot481$, the last giving the subchord which closes on Q.

Observation.—Since stations are of less frequent occurrence in the degree system than in the radius system, it is doubtful if the foregoing procedure is to be preferred to that of using the same chord length and length of spiral, but commencing a full chord at P and closing with a subchord on Q. Here the initial deflection for a $33\frac{1}{3}$ feet chord would be merely multiplied by 1^2, 2^2, 3^2, 4^2, etc., stations being interpolated during progress to deflections given by the squares of the chordages of the station distances multiplied by the initial deflection.

Nevertheless, such procedure cannot economically supersede "chaining through," as given for the radius system.

Spiralling In.—In this method the chord length c is calculated such that k is unity—*i.e.*, $\frac{573c^2}{lr}=1$.

Hence—

$$c=\frac{1}{23\cdot94}\sqrt{RL}\text{ chains, L and R being in chains,} \quad (11)$$

or— $$c=\sqrt{\frac{10L}{D}}\text{ feet, L also being in feet.} \quad (12)$$

Here stations are subsidiary to the construction, and the curve is set out from its commencement with chords of length c, the only subchord occurring at the end of the spiral. Accordingly the deflections run as the squares of the chord number, 1′, 4′, 9′, 16′, etc., the last being $\left(\frac{L}{c}\right)^2$ or $573\,\frac{L}{R}$ or 10LD, which may be made integral in the selection of the length L.

Theod. @	Sighting @												
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	0	1	4	9	16	25	36	49	64	81	100	121	144
1	2	0	4	10	18	28	40	54	70	88	108	130	154
2	8	5	0	7	16	27	40	55	72	91	112	135	160
3	18	14	8	0	10	22	36	52	70	90	112	136	162
4	32	27	20	11	0	13	28	45	64	85	108	133	160
5	50	44	36	26	14	0	16	34	54	76	100	126	154
6	72	65	56	45	32	17	0	19	40	63	88	115	144
7	98	90	80	66	54	38	20	0	22	46	72	100	132
8	126	119	108	95	80	63	44	23	0	25	52	81	112
9	162	152	140	126	110	92	72	50	26	0	28	58	90
10	200	189	176	161	144	125	104	81	56	29	0	31	64
11	242	230	216	200	182	162	140	116	90	62	32	0	34
12	288	275	260	243	224	203	180	115	128	99	68	35	0

The foregoing table affords multipliers for the initial normal deflection k when a point other than the origin

P is occupied, Q being comprehended only in spirals consisting of an integral number of chords. It applies only to spirals commencing with a full chord at P, the case of "spiralling through" requiring individual calculations, q not being integral in $N(3q \pm N)$, where N increases by increments of $\frac{1}{2}$ or $\frac{1}{3}$.

It is evident that the tabular values are the summations forward and rear of tangent points q of arithmetical series with a common difference of 2, the first forward and rear values being respectively $3q+1$ and $3q-1$. Thus, if the instrument is at Point 4, the first forward value is 13, and the first rear value 11; and if, as indicated, *sigma* denotes the instrument at Point 4, the summation of the following series gives the required multipliers, which themselves are deflections in minutes in spiralling in, k being unity.

Spiral Points:	P	1	2	3	4	5	6	7	8	9	10
Differences:		5	+ 7	+ 9	+ *11*	Σ *13*	+ 15	+ 17	+ 19	+ 21	+ 23
Deflections:	32	27	20	11	*0*	13	28	45	64	85	108

Note.—Since the long chord and the tangents at P and Q enclose angles of Ω and 2Ω respectively, it follows that two theodolites may be used, one at P, the other at Q, the deflections from the tangents thereat being respectively N^2k and $N(3L-N)k$, where N is counted in chords or chains from P and Q, and L is the length of transition in chords or chains. This fact suggests the occupation of Q and reversed construction: a process that may be expediently considered in spiralling through and in spiralling in with an integral number of chords.

INTERPOLATING STATIONS.—Stations at even chainages around the spiral may be interpolated in a number of ways, the following being preferable with the theodolite.

(i.) By calculating the *chordage* of the station points and setting out the deflections as the squares of these values.

(ii.) By adding a deflection correction to the preceding deflection, the differences in deflection being regarded proportional to the length of the chord.

Consider these with reference to the case cited on p. 20, a 36 chain curve with spiral approach, the latter leaving the main tangent at 26·82 chains.

Here stations will occur on the curve at the following distances from the point of spiral P—

·18 1·18 2·18 3·18 4·18 5·18,

and if the length of transition is taken as $\sqrt{36}$ chains, the chord length c will be $\frac{1}{23{\cdot}94}\sqrt{36\times 6}$, or ·614 chain.

The chordages of the stations will then be in order—

$$\begin{array}{cccccc} \cdot 18 & 1{\cdot}18 & 2{\cdot}18 & 3{\cdot}18 & 4{\cdot}18 & 5{\cdot}18 \\ \hline \cdot 614 & \cdot 614 & \cdot 614 & \cdot 614 & \cdot 614 & \cdot 614 \end{array}$$

or ·29 1·92 3·55 5·02 6·81 8·44,

the third decimal place being negligible.

(i.) Then the deflections for the instrument at the point of spiral P will be as the squares of these values, viz.:

·08 3·69 12·60 25·20 46·37 71·23.

This process is tantamount to calculating the deflections for the given distances in chains from P, by multiplying the squares of those distances by the deflection angle per chain.

(ii.) Now the differences in deflection per chord run in the series 1, 3, 5, 7, etc., and the corrections are accordingly—

·29×1′ ·92×3′ ·55×7′ ·02×11′ ·81×13′ ·44×17′,

the multipliers being one more than double the integers in the chordages.

Adding these to the squares of the integers in the chordages, the following deflections are obtained:

·29′ 3·76′ 12·85′ 25·22′ 46·53′ 71·48′.

It will be seen that these are approximate values, being too great by the product of the fraction into its difference from unity. In no case, however, will the error exceed 0·25′. But the exact value can be obtained by multiplying the fraction into the sum of twice the integer plus the fraction.

Thus, for chordage 3·55, the exact deflection will be $3^2+\cdot55(2\times3+\cdot55)=9+3\cdot60$, or 12·60 minutes. It is thus more accurate to take the difference into twice the integer when the fraction is under 0·5.

Stations inserted by approximate means should never be occupied in the construction. Besides, when fractional angles occur, it is doubtful whether with coarse instruments stations are located any more accurately than if they are placed on the chord, or offset by some simple rule.

This subject introduces the matter of offsets from the chord.

Offsets from the Chord.—While the angles between successive tangents are in the series 3, 9, 15, etc., the angles between successive chords and their tangents are in the series 1, 4, 7, etc., back values being greater by unity in the series 2, 5, 8, etc. It is therefore evident that the farther the chord is from the origin, the more nearly its arc approaches that of a circle, and, in practice, no appreciable error will be intro-

duced by regarding each arc as circular in ascertaining the offsets from the chord in interpolating stations. Hence, if c be the chord length in chains or 100 feet units, f the fraction of a chord at which a station falls, and a the deflection angle from the tangent at the chord considered, then the offset z will be given by $f(1-f)ca$. But a is initially one minute, or ·00029 radian, its values increasing in the series (3N − 2), which may be written (3I+1), where I is the integer in the chordage. Whence the rule, $z=af(1-f)(3I+1)c$, where a is $\frac{1}{4}''$ or $\frac{3}{8}''$ according as c is in chains or hundreds of feet.

Thus, in application to the 36 chain curve of pp. 20 and 26, where the chordages of the stations are ·29, 1·92, 3·53, 5·02, 6·81, and 8·44, the offsets will be in order $\frac{1}{4}''\times\cdot29(1-\cdot29)(0+1)\cdot614=0$, $\frac{1}{4}''\times\cdot92(1-\cdot92)\times(3+1)\times\cdot614=0$, $\frac{1}{4}''\times\cdot53(1-\cdot53)(9+1)\cdot614=\frac{3}{8}''$, $\frac{1}{4}''\times\cdot02(1-\cdot02)(15+1)\cdot614=\frac{1}{16}''$, $\frac{1}{4}''\times\cdot81(1-\cdot81)(18+1)\cdot614=\frac{7}{16}''$, and $\frac{1}{4}''\times\cdot44(1-\cdot44)(24+1)\cdot614=\frac{15}{16}''$.

II. By Offsets.—In this method the spiral is constructed in the same manner as Froude's cubic parabola. In fact, the data of the clothoid and cubic become identical when abscissæ x along the main tangent are regarded equal to spiral distances, which is a safe assumption for most of the radii occurrent on main lines.

Here $x=m\sqrt{\phi}$ and $y=\frac{m}{3}\phi^{\frac{3}{2}}$ (Eq. viii., p. 16).

Whence $y=\frac{x^3}{6rl}$ (Eq. 7, p. 16).

But $s=\frac{l^2}{24R}$ (Eq. 8, p. 16).

Hence $$y=4s\cdot\frac{x^3}{l^3}, \qquad (13)$$

where $4s$ is Y, the terminal y ordinate.

The method may be effected in various ways, the following being preferable:

1. Measuring the length of transition L along the main tangent from the point of spiral P, and taking the abscissæ x as successive tenths of that length, the y offsets from the main tangent running as the cubes of the intervals, being in order:

$$k \quad 8k \quad 27k \quad 64k \quad 125k \quad 216k \quad 343k \quad 512k \quad 729k \quad 1000k$$

where k is equal to $4s \div 1000$.

Thus if s be 2′ 6″, k will be ·01 foot, and the offsets in inches will be as follows:

$$\tfrac{1}{8}'' \quad \tfrac{5}{16}'' \quad 3\tfrac{1}{4}'' \quad 7\tfrac{11}{16}'' \quad 1'\,3'' \quad 2'\,1\tfrac{15}{16}'' \quad 3'\,5\tfrac{3}{16}'' \quad 5'\,1\tfrac{7}{16}'' \quad 7'\,3\tfrac{1}{2}'' \quad 10'\,0''.$$

When, however, it is desired to introduce the angular dimensions of the clothoid, fixing only P and Q in the original location, the spiral itself may be subsequently inserted by measuring the intervals as chords, and tying the ends of these with the corresponding tangent offsets. Here the shift will bisect the actual length of transition, not the terminal abscissa; but, on the other hand, slight lateral deviations will be introduced, yet never to such an extent as would displace the alignment by an eighth of an inch.

For similar reasons, resort may be made to the following process which is involved in the case of compound curves (and reverse curves by Froude's method).*

2. Measuring on one side of the shift centre R, half the length of transition L along the main tangent,

* See p. 43.

and on the other side half along the shifted curve; then at intervals of one-tenth the length L, offsetting the spiral with ordinates which vary as the cubes of the distances from the beginning and end, P and Q respectively.

Thus, if the rule $L=\sqrt{R}$ be embodied, s being consistently 33 inches, the offsets for any spiral will be as follows:

$\frac{5}{10}$ L from P or Q, $16{\cdot}5'' \left(\frac{5}{5}\right)^3 = 16\frac{1}{2}''$ @ R

$\frac{4}{10}$ L ,, ,, $16{\cdot}5'' \left(\frac{4}{5}\right)^3 = 8\frac{7}{16}''$ @ 4, 6

$\frac{3}{10}$ L ,, ,, $16{\cdot}5'' \left(\frac{3}{5}\right)^3 = 3\frac{9}{16}''$ @ 3, 7

$\frac{2}{10}$ L ,, ,, $16{\cdot}5'' \left(\frac{2}{5}\right)^3 = 1\frac{1}{16}''$ @ 2, 8

$\frac{1}{10}$ L ,, ,, $16{\cdot}5'' \left(\frac{1}{5}\right)^3 = \frac{1}{8}''$ @ 1, 9.

. . . . . 0 @ P and Q . . . . . .

It will be seen that both offsets at the shift centre R are equal to half the shift s.

Although the preceding method is theoretically the most exact, the results compare unfavourably with those derived entirely by tangent offsets when the points located on the shifted curve are not abreast of the spiral points. (See p. 64.)

Conclusion.—Normally, four elements are necessary in fixing the limits of the spiral, these being the same whatever data be utilised in the construction: (1) The length L; (2) the shift $s=\frac{L^2}{24R}$; (3) the shift increment,

$s \tan \frac{1}{2}\Delta$ (aS, Fig. 1); (4) the spiral extension $X - R \sin \Phi$ (SP, Fig. 1), the last two together forming the tangent excess aP.

It is desirable, however, that tangent distances should be expressed in terms of the radius, the length, and the shift. Now, it follows that— (Eq. v., p. 15.)

$$X = L\left(1 - \frac{\Phi^2}{10} + \ldots\right); \text{ and } \Phi = \frac{L}{2R}$$

$$= L\left(1 - \frac{L^2}{40R^2} + \ldots\right); \text{ but } s = \frac{L^2}{24R}$$

$$= L\left(1 - \frac{3}{5}\frac{s}{R}\right).$$

Further, the spiral extension $SP = X - R \sin \Phi$; and on expanding $\sin \Phi$ and substituting $\frac{L}{2R}$ for Φ, the third and higher terms being neglected—

$$SP = X - \frac{L}{2}\left(1 - \frac{L^2}{24R^2}\right)$$

$$X - \frac{L}{2}\left(1 - \frac{s}{R}\right).$$

Hence, on substituting for X—

(4) $$SP = \frac{L}{2}\left(1 - \frac{1}{5}\frac{s}{R}\right),$$

or $\frac{1}{2}$ L when R is over 25 chains.

(5) Thus the tangent excess in the fundamental case—

$$a\text{P} = s \tan \tfrac{1}{2}\Delta + \frac{L}{2}\left(1 - \frac{1}{5}\frac{s}{R}\right);$$

(6) And the total tangent distance IP,

$$T_s = (R + s) \tan \tfrac{1}{2}\Delta + \frac{L}{2}\left(1 - \frac{1}{5}\frac{s}{R}\right).$$

The remaining elements of pp. 5 and 6 may be added, being often desirable either as checks or as means of modification.

(7) The terminal x abscissa, $X=L\left(1=\frac{3}{5}\frac{s}{R}\right)$.

(8) The terminal y ordinate, $Y=4\,s=\frac{L^2}{6R}$.

(9) The whole chord PQ and its deflection Ω, or $\frac{1}{3}\,\Phi$, where Φ is the total angle of the spiral.

$$PQ=X \sec \Omega.$$

TABLES.—Tables I. and II. of the Appendix give the elements of spiral approaches to main curves of radii 20 to 80 chains and degrees 1° to 6°. In both, time approach lengths are assigned, those of the former table being calculated from the rule $L=\sqrt{R}$ chains, and those of the latter from $L'=600\div\sqrt{D^\circ}$ feet. The data both for *spiralling through* and *spiralling in* are added, these being respectively the deflections k per chain^2 or per 100 feet^2, and the chord lengths c per minute^2 of deflection. Incidentally it may be noted that the deflections for a spiral of ten integral chords may be found by taking one-hundredth part of the total deflection angle Ω, the corresponding chord lengths being one-tenth of the tabular values of the length L or L′, as the case may be.

After 55 chains radius in Table I., the product LR is constantly equated to 414 in accordance with the rule $L=\frac{V^3}{1381R}$ chains, V being limited to about 83 miles per hour; and, in consequence, the chord c and the deflection k per chain^2 become constant, while the shift s decreases in accordance with the rule $7141\div R^3$ chains. Similarly above 1° 30′ in Table II., $L'=330D^\circ$, in accordance with the rule $L'=\cdot6\left(\frac{V}{10}\right)^3 D^\circ$ feet, but is taken as $333\cdot33D^\circ$ feet

for simplicity, k' and c' becoming constant, and s accordingly reducing to $\cdot 808D^3$ feet.

Finally, it should be remembered that there is considerable latitude in selecting the length of transition; and, in consequence, the tabular values may be regarded as being tentative, a little modification often leading to considerable simplification in the data. For instance, in "spiralling through" to a 27 chain curve, the deflection k per chain^2 could be taken as 4 minutes, giving the length L as 5·315 chains, the total deflection Ω as 112·80 minutes, and the shift s as ·0436 chain, or 2′ 10½″. Similarly, in "spiralling in" to a 36 chain curve, the chord c could be taken as ·628 chain, giving the length L as $10c$, Ω as 100 minutes, and s accordingly ·0457 chain, or 3′ 0⅛″. In fact, a system of integral-chorded spirals may be developed with the basis of 20″, 40″, and 60″ as the initial deflections in spiralling in, the whole embracing all practical requirements without introducing lengths of transition that differ appreciably from the lengths that would otherwise be employed. Such a system would not preclude the use of instruments reading to one minute, division to 20″ being desirable; for in all tabular spirals there are fractions over ½′, which are necessarily dropped in construction with minute reading theodolites.

Notation.—Before passing to the subject of Alignment, it will be advisable to adopt some explicit notation identifying the limits of spirals, the points P and Q of the first spiral becoming P′ and Q′ in the second spiral of the same curve (Fig. 1). But in progress from P to P′, the *end* Q′ of the second spiral becomes the constructional *beginning*, while P′ becomes likewise the constructional *end*. Hence the

exclusive adoption of the following terms and abbreviations:

P: *Point-Spiral, P.S.*
Q: *Point-Spiral-Curve, P.S.C.*
Q′: *Point-Curve-Spiral, P.C.S.*
P′: *Point-Spiral-Tangent, P.S.T.*

Similarly, in compound curves, the *point* of *compound curvature, P.C.C.*, may be superseded thus—

C: *Point* of *Spiral Compound, P.S.S.*;

and, in reverse curves, the *point* of *reversed curvature, P.R.C.*, thus—

R: *Point* of *Spiral Reverse, P.S.R.*,

the tangent points of circular arcs being designated as follows: *a* or A, *Point* of *Curve, P.C.*; *a′* or A′, *Point* of *Tangent, P.T.*

PART II

ALIGNMENT

THIS section deals with the various problems of alignment that arise when transition curves are introduced, either in the original location or in easing existing lines, or spiralling old track, as the Americans style it.

The insertion of transition curves, or spirals, may be said to introduce four general problems of alignment, the following :

1. *Shifting the main curves so as to admit the transition curves without change of radii.*

2. *Assuming the original radius to embody the shift and so preserve the conditions of tangency.**

3. *Sharpening the radii of the main curves so as to preserve the original location as nearly as possible.*

4. *Changing the radii of the main curves so as to preserve as nearly as may be the original length of the line.*

As a rule, straights are not shifted and, in consequence, the following is not considered—

5. Shifting the straights so as to admit the transition curves without disturbing the main curves.

Nevertheless, it must be noted that the improvement of existing lines may allow such latitude as will relax

* The second of these may be regarded as affording another aspect of the first, and, in consequence, will be considered with that case.

the conditions imposed in the foregoing problems; and, in this connection generally, it may be added that only the constructional lines are regarded in the cases treated, the artifices with which the engineer would transfer to or from such alignment being a matter for his ingenuity.

General Problems.—A fitting introduction to the present subject is afforded by the following construction, which may be regarded as the general case of alignment. (Fig. 3)

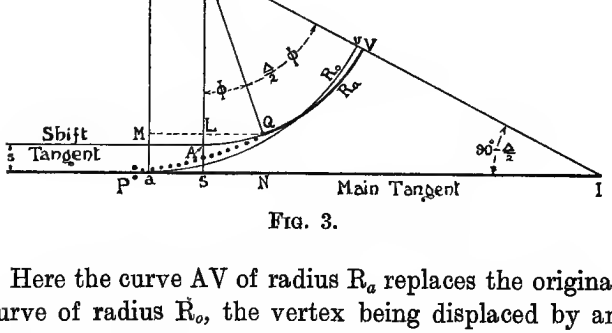


FIG. 3.

Here the curve AV of radius R_a replaces the original curve of radius R_o, the vertex being displaced by an amount h from v to V.

Then $ao = NQ + MH + Ho$.

But $ao = R_o$; $NQ = Y$, $MH = QO \cos \Phi = R_a \cos \Phi$;

$$oH = oO \cos \tfrac{1}{2}\Delta = (h + R_o - R_a) \cos \frac{\Delta}{2}.$$

$$\text{Hence } R_a = \frac{R_o \text{ vers } \frac{1}{2}\Delta}{\cos \Phi - \cos \frac{\Delta}{2}} - \frac{h \cos \frac{\Delta}{2} + Y}{\cos \Phi - \cos \frac{\Delta}{2}}. \qquad (1)$$

And $h \cos \frac{\Delta}{2} = (R_o - R_a) \text{ vers } \frac{\Delta}{2} + R_a \text{ vers } \Phi - Y$. (2)

But s, the shift $= QN - LA = Y - R_a \text{ vers } \Phi$.

Hence $h \cos \frac{\Delta}{2} = (R_o - R_a) \text{ vers } \frac{\Delta}{2} - s$. (3)

When $h = 0$, $s = (R_o - R_a) \text{ vers } \frac{1}{2}\Delta$. (See p. 44.)

When $R_o = R_a$, $h = s \sec \frac{\Delta}{2}$, being plus since it is measured in the opposite direction.

Also, the tangent excess,

$$Pa = X - R_a \sin \Phi - (h + R_o - R_a) \sin \tfrac{1}{2}\Delta. \quad (4)$$

Or, substituting for h in terms of the shift,

$$Pa = X - R_a \sin \Phi - (R_o - R_a) \tan \tfrac{1}{2}\Delta + s \tan \tfrac{1}{2}\Delta. \quad (5)$$

When $h = 0$, $Pa = X - R_a \sin \Phi - (R_o - R_a) \sin \frac{1}{2}\Delta$. (See p. 44.)

When $R_o = R_a$, $Pa = X - R_a \sin \Phi + s \tan \frac{1}{2}\Delta$. (See p. 5.)

The spiral extension $X - R_a \sin \Phi$ has been shown to be equal to $\frac{1}{2} L\left(1 - \frac{1}{5} \frac{s}{R_a}\right)$, or $\frac{L}{2}$ when R_a is over 25 chains. (See p. 31.)

Compound Curves.—At this juncture it will be convenient to note that two cases arise in connection with compound curves, viz.:

(*a*) When spirals are to be inserted between the branches of the compound, and also at the beginning and the end of the composite curve.

(*b*) When spirals are to be inserted only at the beginning and the end of the compound curve.

This distinction introduces that important matter, the difference in the radii R and r that requires a transition between the branches of the compound.

Now it may be shown that, as in Froude's parabola, the shift s_c for the compound is the difference of the shifts S and s of the branches, the relation being expressed algebraically as

$$s_c = \frac{l_c^2}{24}\left(\frac{R - r}{Rr}\right), \tag{6}$$

where R and r are the radii and l_c the length of transition connecting the shifted curves.

It is evident that such transition will be necessary only when the radii R and r differ materially.

Assume the curves to be placed so that they yield the requisite shift along a common radial, R being greater than r.

Now, if e be the superelevation required by the curve of radius r, and E that required by the curve of radius R, the latter being the lesser value under the controlling speed of the sharper curve; then, since the superelevation of the sharper curve when run out on to the straight should be run out in the same time as into the flatter curve, the velocity v being constant—

Elev. e lost in $\frac{l_1}{v}$ secs. = Elev. $(e - E)$ lost in $\frac{l_c}{v}$ secs.,

where l_1 and l_c are the respective lengths of transition to straight and to greater circle.

Hence $$\frac{l_c}{l_1} = \frac{e - E}{e} = \frac{M\left(\frac{1}{r} - \frac{1}{R}\right)}{\frac{M}{1}{r}} = \frac{R - r}{R},$$

M being equal to $\frac{GV^2}{990}$. (Page 2)

Whence $$l_c = l_1\left(1 - \frac{r}{R}\right). \tag{7}$$

Mr. Shortt arrives at the same result by equating the rate of change in acceleration in entering the smaller curve from the straight to the rate of change in acceleration in entering from the larger radius, the lengths of transition remaining l_1 and l_c respectively. That is—

$$\frac{v^2}{r} \div \frac{l_1}{v} = v^2\left(\frac{1}{r} - \frac{1}{R}\right) \div \frac{l_c}{v}.$$

Further, assuming $\sqrt{r}$ to be the length of the transition l_1 in chains, $l_c = \left(1 - \frac{r}{R}\right)\sqrt{r}$; and, on substituting for l_c in (6),

$$s_c = \frac{1}{24}\left(1 - \frac{r}{R}\right)^3 \text{ chains.} \qquad (8)$$

Since a track cannot be lined to closer than $\frac{1}{8}$th inch, this value of s_c may be said to determine the limit beneath which it is impossible to insert a transition curve between the branches.

Hence, if s_c be put equal to $\frac{1}{6336}$ chain, $\left(1 - \frac{r}{R}\right) = \sqrt[3]{\frac{1}{264}}$, and $\frac{r}{R} = 85$ per cent. very nearly. But with considerably larger shifts than this the curves would be practically identical with the transition; so if $s_c = 3''$ be fixed as the limit at which the latter is introduced, the difference in the radii R and r may be as much as 40 per cent. However, a lower difference of radii may call for a transition curve, and its inefficient length may be by no means derogatory to the alignment.

1. *Shifting the main curves so as to admit the transition curves without change of radii.*

(i.) *Simple Curves.*—This case was considered in determining the elements of transition curves, the

shift s being generally $Y - R \sin \Phi$, or $\frac{L^2}{24R}$ in the case of the clothoid and cubic. Also, in general, the tangent excess $Pa = X - R \sin \Phi + s \tan \frac{1}{2}\Delta$, which reduces to $\frac{L}{2}\left(1 - \frac{1}{5}\frac{s}{R}\right) + s \tan \frac{1}{2}\Delta$, or $\frac{1}{2}L + s \tan \frac{1}{2}\Delta$ very nearly.

Here it should be noted that if the original radius R_o is taken such that $R_o = R_a + s$, the shift term $s \tan \frac{1}{2}\Delta$ disappears, and the remainder, the spiral extension $\frac{L}{2}\left(1 - \frac{1}{5}\frac{s}{R}\right)$, is measured from the original point of tangency (a, Fig. 1).

(ii.) *Compound Curves.*—(a) When the spirals are to be inserted between the branches of the compound, and also at the beginning and the end of the composite curve.

When the transitions are inserted in the original location it may be possible to retain the conditions of tangency by assuming radii $R + S$ and $r + s$, where R and r are the given radii, and S and s the end shifts accordingly, the difference of which is the central shift, or gap d. (Fig. 4)

Otherwise, as is the more usual, d will be equal to $s \sec \beta - S \sec a$, the curves moving along a common radial to a and β as central angles.

Now in either case, it seldom happens that d is equal to s_c, the value required by $\frac{l_c^2}{24}\left(\frac{R - r}{Rr}\right)$, and the question arises as to whether the central transition be omitted or not, and in what manner must the entire construction be modified. This introduces the following artifice, which especially applies when the com-

pound curves have been run in without provision for transition curves.

Consider the case when the shifts have been calculated for transition to the straights from curves of radii R and r:

$$\text{Gap } C_2C_1 = d = C_2c - C_1c = s \sec \beta - S \sec \alpha. \quad \text{(i.)}$$

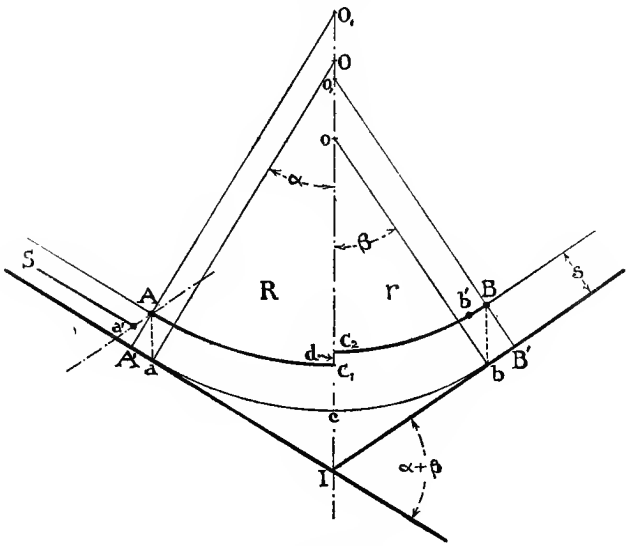


FIG. 4.

CAUTION.—Now if time approaches be adopted, it is not to be inferred that $S = s$, and $d = s$ (sec β—sec α) consequently; for the speed of the sharper radius will obtain, and will thus reduce the length L and the shift S accordingly.

As a rule the gap d will not accord with the correct shift s_c, but will be much greater. However, considerable latitude is to be countenanced rather than resort

to modification of data or, particularly, shifting the end transitions in the manner hereafter described. Often, however, a central transition corresponding to the gap d would be incongruous, and the latter operation becomes inevitable.

Therefore, let a suitable length l_c be assumed and the corresponding shift s_c be calculated, the sharper curve bc alone being shifted.

Then $$cC_1 = cC_2 - s_c = s \sec \beta - s_c. \qquad \text{(ii.)}$$

Now imagine the curve ac shifted inwards until c comes to C_1 and a moves to A.

Then

$$AA' = aA \cos a = C_1c \cos a = (s \sec \beta - s_c) \cos a. \qquad \text{(iii.)}$$

It is unlikely that AA′ will be equal to the end shift S. Therefore if AA′ is greater than S, the point B must be slid along the tangent to b' by an amount Bb', which is equal to Aa', the latter distance being along a tangent parallel to bI.

Now the angle $a'AA' = 90° - (a+\beta)$; and $Aa' = (AA' - S) \sec a'AA'$.

Hence $$a'A = \frac{(s \sec \beta - s_c) \cos a - S}{\sin (a+\beta)}. \qquad (9)$$

Had it been necessary to pull back B, AA′ being less than S, the foregoing distance would have been

$$\frac{S - (s \sec \beta + s_c) \cos a}{\sin (a+\beta)}. \qquad (10)$$

(*b*) When spirals are to be inserted only at the beginning and the end of the compound curve.

Here the gap C_2C_1 becomes zero, and S sec $a = s \sec \beta$. But, as is so often the case, S and s do not

together accord with suitable lengths of transition; and unless modification of length is permissible, the compound must be shifted by the foregoing process, s or S being taken at its correct value. The shift along the tangent in this case will be—

$$\frac{S(\sec\beta\cos\alpha - 1)}{\sin(\alpha+\beta)}, \tag{11}$$

or

$$\frac{S(1 - \sec\beta\cos\alpha)}{\sin(\alpha+\beta)}, \tag{12}$$

according as AA′ is less or greater than S.

(iii.) *Reverse Curves.*—The transition between the branches of a true reverse should consist of two separate spirals, the superelevation changing from zero to the opposite rails at the junctions of these.* Hence the first and second cases should be reduced to those of simple curves, the radii being sharpened so as to admit suitable lengths of transition through the medium of a separating piece of straight. Here the simplest construction is that in which the centres of curvature are moved inwards along the radials at the original tangent points, the latter points remaining unchanged. As a rule the central angles Δ and δ will be equal, or the reverse will have a common radius R_o. Even then the process is somewhat complex, since it involves finding an auxiliary angle x, an angle θ, and thence the

* Froude's cubic reverse is certainly an exception, not being a true transition reverse, but a curve in which the point of spiral reverse R is virtually a point of inflexion, outside which the radius of curvature is double that of similar reversed circular curves. Such is constructed in the manner outlined for simple curves, cubic offsets being erected from the redundant portion of *each* circular curve. (See p. 29.)

reduced central angles a or β, or both, the trigonometrical relations giving the inner tangent points of the reduced circular curves. The calculations are simplified by embodying the shift in the reduced radii, thus eliminating the shift term $s \tan \frac{1}{2}a$ or $s \tan \frac{1}{2}\beta$, or both.

3. *Sharpening the radii of the main curves so as to preserve the original location as nearly as possible.*

(*a*) Consider first the case of the vertex remaining unaltered, the entire curve being superseded by one of reduced radius.

(i.) *Simple Curves.*—Let R_o and R_a denote the original and amended radii respectively, a shift tangent being drawn parallel to the main tangent at A. (Fig. 5)

Then $$R_a + s = R_o - on.$$

But $on = (R_o - R_a) \cos \frac{\Delta}{2}$, where Δ is the intersection or central angle.

Hence $$s = (R_o - R_a) \text{ vers } \frac{\Delta}{2}. \qquad (13)$$

Also the distance $Pa =$

$$X - R_a \sin \Phi - (R_o - R_a) \sin \tfrac{1}{2}\Delta. \qquad (14)$$

The foregoing result may be derived by substituting $h = 0$ in Eq. 5 of the General Construction (p. 37).

(*b*) As a rule, however, the problem will consist in compounding the portions of the main curve adjacent to the transition curves, thus leaving the central portion of the main curves undisturbed. Here V becomes the point of compound curvature, and $\frac{\Delta}{2}$ the angle γ cut of the curve of radius R_o and replaced by the R_a curve.

Hence $$s=(R_o - R_a) \text{ vers } \gamma. \quad (15)$$

Also, in general, the distance $Pa=$

$$X - R_a \sin \Phi - (R_o - R_a) \sin \gamma; \quad (16)$$

or, in terms of the shift and length,

$$\tfrac{1}{2}L\left(1 - \frac{1}{5}\frac{s}{R_a}\right) - (R_o - R_a) \sin \gamma. \quad (17)$$

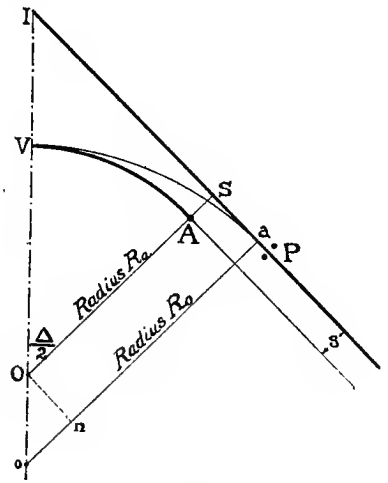


FIG. 5.

The angle γ must be less than $\frac{\Delta}{2}$; in fact, it should not exceed $\frac{\Delta}{4}$ in practice.

(ii.) *Compound Curves.*—(*a*) In this connection the radius of each arc may be shortened, giving two new arcs compounded at the same point V (Fig. 5), the central angles becoming α and β corresponding to radii R_o and r_o, which reduce to R_a and r_a respectively.

Hence by analogy with Eq. 13,

$$\left.\begin{aligned} S&=(R_o-R_a)\text{ vers }a\\ s&=(r_o-r_a)\text{ vers }\beta \end{aligned}\right\} \tag{18}$$

And

$$\left.\begin{aligned} Pa&=X-R_a\sin\Phi-(R_o-R_a)\sin a\\ P'a'&=x-r_a\sin\phi-(r_o-r_a)\sin\beta \end{aligned}\right\} \tag{19}$$

(*b*) But either arc may be compounded again, leaving the central portion undisturbed. This is merely an extension of the case of a simple curve. (See p. 44.)

(iii.) *Reverse Curves.*—Here the reverse branches will be sharpened so that the point of spiral reverse R as a point of spiral P coincides with the original point of reverse curvature. That is, by Eq. 5 of the General Construction, h being 0, and Pa also 0,

$$X-R_a\sin\Phi=(R_o-R_a)\sin\frac{\Delta}{2}. \tag{20}$$

Since $X-R_a\sin\Phi=\frac{1}{2}L=\frac{a}{2}\sqrt{R_a}$ when R_a is over 25 chains and D is under 4°,

$$R_a+\frac{a\sqrt{R_a}}{2\sin\frac{1}{2}\Delta}-R_o=0, \tag{21}$$

where Δ is the central angle of the branch considered, a being *unity* or *eight* according as the radii are expressed in chains or feet.*

The respective solutions of the foregoing quadratic are—

$$\sqrt{R_a}=-\frac{1}{4\sin\frac{1}{2}\Delta}\pm\sqrt{R_o+\frac{1}{16\sin^2\frac{1}{2}\Delta}}\ \textit{chains}. \tag{21a}$$

$$\sqrt{R_a}=-\frac{2}{\sin\frac{1}{2}\Delta}\pm\sqrt{R_o+\frac{4}{\sin^2\frac{1}{2}\Delta}}\ \textit{feet}. \tag{21b}$$

* This assumption increases the shift in the degree system from 2·618 to 2·667 feet, or 2 feet 8 inches.

Also from Fig. 5, $aS = R_o \tan \frac{1}{2}\Delta - (R_a + s) \tan \frac{1}{2}\Delta$

$$= (R_o - R_a) \sin \tfrac{1}{2}\Delta. \qquad (22)$$

It is evident that the foregoing equations give the necessary length of transition for each portion of the reverse, and the same lengths occur where the ends of the reverse join the straights.

4. *Changing the radii so as to preserve as nearly as may be the original length of the line.*

(i.) *Simple Curves.*—The expression leading to this result in the case of a simple curve reduces to—

$$h \cos \tfrac{1}{2}\Delta = (R_o - R_a) \operatorname{vers} \tfrac{1}{2}\Delta - s, \qquad \text{(Eq. 3, p. 37)}$$

where s is the shift and h the displacement of the vertex along the centre line.

Accordingly, the expression for the tangent excess is

$$Pa = X - R_a \sin \Phi - (R_o - R_a) \tan \tfrac{1}{2}\Delta + s \tan \tfrac{1}{2}\Delta. \qquad \text{(Eq. 5, p. 37)}$$

It is often possible to insert a number of transition curves so that the total distance between common points is unaltered, the relation between the original and amended half-lengths being generally—

$$X - R_a \sin \Phi - (R_o - R_a) \tan \tfrac{1}{2}\Delta + s \tan \tfrac{1}{2}\Delta + R_o \frac{\Delta}{2}$$

$$= L + R_a \left(\frac{\Delta}{2} - \Phi \right), \qquad (23)$$

where L is the length of transition.

But it is only by introducing the displacement h that the same length of line can be obtained in any individual curve when inserting transition curves.

Let the original main curve of radius R_o be super-

seded by one of radius R_a, the latter's vertex v being displaced centrally.

Then, since the *original* and *amended* half-lengths must be equal,

$$\underbrace{X - R_a \sin \Phi - (R_o - R_a) \tan \tfrac{1}{2}\Delta + s \tan \tfrac{1}{2}\Delta + R_o \frac{\Delta}{2}}_{\text{Original half-length}} = \underbrace{R_a\left(\frac{\Delta}{2} + \Phi\right)}_{\text{Amended half-length,}}$$

the length of the clothoid being $2R_a\Phi$.

Now, $X - R_a \sin \Phi$ is $\frac{1}{2}L$ very nearly—*i.e.*, $R_a\Phi$.

Hence

$$R_a\left(\frac{\Delta}{2} + \Phi\right) = R_a\Phi - (R_o - R_a) \tan \tfrac{1}{2}\Delta + s \tan \tfrac{1}{2}\Delta + R_o \frac{\Delta}{2}$$

which leads to the *Equation* of *Equal Routes*—viz.,

$$s \tan \tfrac{1}{2}\Delta = (R_o - R_a)(\tan \tfrac{1}{2}\Delta - \tfrac{1}{2}\Delta). \qquad (24)$$

Therefore, if s be assumed, R_a is determinate. But s is constant in time approach, being normally 33 inches or 31·42 inches by the rules given on pp. 11 and 12.

(ii.) *Compound Curves.*—Compound curves without central transitions may be dealt with by an analogous process. Here Eq. 3 becomes $h \cos \alpha = (R_o - R_a) \operatorname{vers} \alpha - S$, for the larger radius, and $h \cos \beta = (r_o - r_a) \operatorname{vers} \beta - s$, for the shorter radius, where S and s are the shifts, R_o and r_o the original radii, and α and β the central angles.

But since the central displacement must have the same value h in each case—

$$\frac{(R_o - R_a) \operatorname{vers} \alpha - S}{\cos \alpha} = \frac{(r_o - r_a) \operatorname{vers} \beta - s}{\cos \beta} \qquad (25)$$

If time approaches are considered, the shift s for the sharper radius will be known, leaving as unknowns R_a, r_a, and S. It will therefore be necessary to determine the correct values of these by trial, assuming R_a and r_a, and determining S for the radius R_a under the controlling speed of the radius r_a.

Another application of the general equation may be to determine dimensions when the displacement h is limited, the equalising of routes being a secondary consideration. In this, Eq. 3 will contain two unknowns, h and r_a, s being constant in time approaches (pp. 37 and 48). Hence, if the vertex displacement h is known, then is the reduced radius r_a determinate. But this will be evident in the succeeding case, where the problem is extended to reverse curves.

However, if the Equation of Equal Routes be embodied, the assuming of h, r_a, or R_a will be precluded; for then will $s=\frac{2}{3}(r_o - r_a)$ vers β, $S=\frac{2}{3}(R_o - R_a)$ vers α, and $s \cos \alpha = S \cos \beta$, s being known by the controlling speed of the sharper branch.

(iii.) *Reverse Curves.*—In the case of the true reverse, the point of reversed curvature becomes the common point of spiral of two separate amended curves which overlap the original ones. The tangent excess, therefore, will be the same at opposite ends of each curve, being zero in Eq. 4. (See p. 37.)

Thus:

$$X - R_a \sin \Phi - (h + R_o + R_a) \sin \tfrac{1}{2}\Delta = 0, \qquad (26)$$

where Δ is the central angle and R_o the original radius of either branch.

Therefore, since $X - R_a \sin \Phi = \frac{1}{2}L$ very nearly, and $L = a\sqrt{R_a}$ in time approach,

$$R_a + \frac{a\sqrt{R_a}}{2 \sin \frac{1}{2}\Delta} - (R_o + h) = 0, \qquad (27)$$

a being *unity* or *eight* according as R_a is expressed in chains or feet.

Whence, if h be assumed, R_a is determinate.

Further, from Eq. 5, h being unlimited and time approaches being assumed—

$$r_a + \frac{a\sqrt{r_a}}{2 \tan \frac{1}{2}\Delta} - (r_o - s) = 0, \qquad (28)$$

a being *unity* or *eight* according as r_a is expressed in chains or feet.

Here s has its normal value, and the consequent displacement of the vertex is expressed by the relation

$$h \cos \tfrac{1}{2}\Delta = (r_o - r_a) \operatorname{vers} \frac{\Delta}{2} - s. \qquad \text{(Eq. 2, p. 37)}$$

If the radii differ, R_o being that of the flatter branch, the latter's amended value R_a will have to be determined, leading to a length L and a shift S to the speed of the sharper curve. But in sharpening generally from equal original radii, it seldom happens that R_a will be equal to r_a; and, in practice, it would be advisable to adopt throughout the length of transition l of the sharper amended branch. In fact, existing conditions would seldom permit transitions in accordance with the rule $a\sqrt{r_a}$, and, in consequence, reduction to $\frac{2}{3}\, a\sqrt{r_a}$ may be necessary.

PART III

FIELD EXERCISES

Problem 1.—The pickets indicate two intersecting straights which are to be connected by a 30 chain circular curve with transition curves, the highest working speed being the maximum for the radius.

Continue the pegging out of the centre line from 84·00 chains, the peg last driven "chaining through," and inserting additional pegs at half-chain intervals on the spirals.

PROCEDURE.—1. Set up the theodolite over the point of intersection I, and measure the angle between the straights supplementary to the intersection angle Δ.

For purposes of calculation, let the angle Δ be 56° 20′.

2. Calculate IP, IP′, the total tangent distances $T_s=(R+s)\tan\frac{1}{2}\Delta+\frac{1}{2}L$, where L is the length of transition in chains and s the shift. But, since the maximum speed $11\sqrt{R}$ is permissible, L will be $\sqrt{R}$ chains in time approach, and s 33 inches accordingly.

Thus: T_s will be $(30+\frac{1}{24})\tan 28°\,10'+\frac{1}{2}\sqrt{30}=$
$(30+\frac{1}{24})\times\cdot53545+2\cdot739=18\cdot829$ chains.

3. Measure the total tangent distances IP, IP′, and drive three pegs, two guarding the centre peg at the points of spiral P, P′. Continue the chaining forward, and record the chainage of the peg P.

Let chainage 84·00 be ·72 chain short of P; then the

chainage of P will be 84·72, and the chainage of Q 90·197.

4. Calculate the remaining elements of the spirals—viz., (*a*) the Long Chords PQ and P′Q′, and the Total Deflection Angle Ω, also (*b*) the Deflection per chain² of the spirals, and (*c*) the Tangential Angle per chain of the circular portion. Complete the data of the composite curve, finding the chainages of Q′ and P′, and tabulating the deflection and tangential angles for the pegs on the curves.

(*a*) Total deflection angle $\Omega = 573\,\frac{L}{R} = 1°\ 44{\cdot}61'$, and Φ, the total spiral angle = 5° 13·83′.

Long chords PQ, $P'Q' = X \sec \Omega = L\left(1 - \frac{3}{5}\,\frac{s}{R}\right) \sec 1°\ 45' =$ 5·473 chains.

(*b*) Deflection k per chain² $= \frac{573}{LR} = 3{\cdot}507$ minutes.

(*c*) Tangential angle per chain $\sin^{-1}\frac{1}{2R}$ or $\frac{1719}{R}$ minutes (nearly) = 57·3 minutes.

Length of circular portion $= \frac{(\triangle - 2\Phi)}{2\delta} = \frac{45°\ 52{\cdot}3'}{1°\ 54{\cdot}6'} = 24{\cdot}017$ chains.

Hence chainage of Q′ = 114·214; of P′ = 119·691.

Deflection angles from tangent at P in minutes:

(·28)² × *3·507*; (·78)² × 3·507; (*1·28*)² × *3·507*; (1·78)² × 3·507; (*2·28*)² × *3·507*; (2·78)² × 3·507; (*3·28*)² × *3·507*; (3·78)² × 3·507; (*4·28*)² × *3·507*; (4·78)² × 3·507; (*5·28*)² × *3·507*.

Deflection angles from tangent at P′ in minutes:

(·191)² × 3·507 (*·691*)² × *3·507*; (1·191)² × 3·507; (*1·691*)² × *3·507*; (2·191)² × 3·507; (*2·691*)² × *3·507*; (3·191)² × 3·507; (*3·691*)² × *3·507*; (4·191)² × 3·507; (*4·691*)² × *3·507*; (5·191)² × 3·507.

Tangential angles from tangent at Q in minutes:
57·3′ × ·803; 57·3′ × 1·803; 57·3′ × 2·803; 57·3′ × 3·803;
57·3′ × 4·803; 57·3′ × 5·803 . . . 57·3′ × 23·803.

5. Set up the theodolite over the *point-spiral* P, clamp the plates at 0°, and sight the *point* of *intersection* I. Backsighting thus, set off the several deflection angles, and direct the chainmen accordingly, giving first the deflection for the subchord, then those for

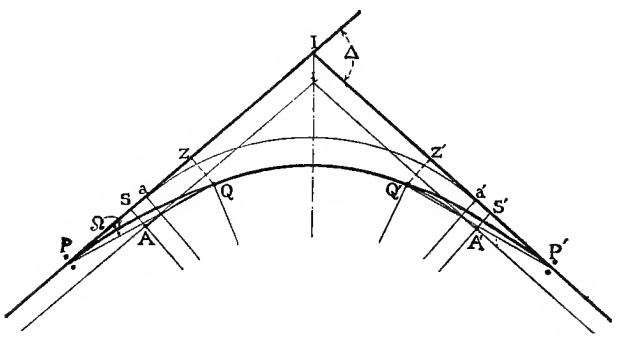


Fig. 6.

successive half-chain chords, and finally that for the subchord closing on Q. Check the peg at Q by measuring the long chord PQ along the direction given finally by the theodolite, the circle remaining clamped at the total deflection. As a further check, note that the perpendicular distance from Q to the tangent should be 4*s*, or 132 inches.

6. Set the theodolite over the *point-spiral-curve* Q; clamp the plates at twice Ω, and, sighting P, lay down a tangent at Q by transiting the telescope and bringing the plates to 0°. Proceed to run in the circular

portion QQ′, giving the tangential angles and directing the chainmen accordingly in laying down, first the subchord, then successive chain chords, and finally the subchord closing on Q′. Check the position of Q′ by measuring the terminal ordinate N′Q′, which should be equal to 4*s*. (See Note vi.)

7. Set up the theodolite over the *point-spiral-tangent* P′, clamp the plates at 0°, and sight the point of intersection I. Backsighting thus, lay off an angle Ω, and observe if the cross hairs of the telescope bisect the point-curve-spiral Q′. Then set off the tabulated deflections in descending order, the first being the value under Ω. At the same time direct the chainmen to measure accordingly, first the subchord from Q′, then successive half-chain chords, and finally the subchord closing on P′. (See Note vii.)

Note vii.—In "spiralling through," the second spiral Q′P′ may be set out from Q′ with little additional calculation, the deflections from the tangents at P and P′ being invariably different.

Here the chainages of spiral points from Q′ are ·786, 1·286, 1·786, etc., for N, while q is the length L, 5·477 chains, in $N(3q - N)k$, k being the deflection per chain, or 3·507 minutes.

The chief difficulty, however, is that of exactly locating Q′, since when possible to measure the whole chord QQ′ the error of measurement is likely to be appreciable. Nevertheless, when locating Q′ from Q with the whole tangential angle $\frac{1}{2}(\Delta - 2\Phi)$, arrows could be inserted in the line QQ′, one on each side of the located curve position of Q′, and the instrument could be adjusted in this line until an angle 2Ω from the tangent at Q′ bisects the peg at P′, the tangent at Q′ being obtained by turning off from Q

an angle $\frac{1}{2}(\triangle - 2\Phi)$. The procedure of Problem 4 is detailed so as to demonstrate the various artifices that might be resorted to when working progressively through the ruling points P, Q, C, Q′, P′, of a compound curve.

OBSERVATIONS.

Problem 2.—Connect the given straights with a circular curve of 36 chains radius, introducing suitable spiral approaches in the original location.

(Assuming the chainage of the point of spiral P to be 76·32, interpolate pegs at even chainage on spirals and curve.)

PROCEDURE.—1. Set up the theodolite over the *point* of *intersection* I and measure between the straights the angle supplementary to the intersection angle Δ. (See Fig. 6.)

For purposes of calculation, let the angle Δ be 52° 30′.

2. Calculate the elements and data of the spirals, assuming construction by deflections of one minute per chord².

On inspecting the table on p. 100, it will be seen that the total deflection $\Omega = 95{\cdot}5$ minutes, whereas $\Omega = 100{\cdot}0$ minutes would give *ten* integral chords without unduly increasing the length L, which is normally 6·000 chains.

(*a*) Length $L = \frac{R\Omega}{573} = \frac{3600}{573} = 6{\cdot}283$ chains, or 28 links in excess.

Adopting this length L, the normal shift s will be increased from ·0417 chain to ·0457 chain by the ratio $\frac{(6{\cdot}283)^2}{(6{\cdot}000)^2}$.

(*b*) Total tangent distances Ts

$$= (R+s)\tan \tfrac{1}{2}\Delta + \tfrac{1}{2}L\left(1 - \frac{1}{5}\frac{s}{R}\right)$$

$$= (36 + {\cdot}0457)\tan 26° 15' + 3{\cdot}142\left(1 - \frac{1}{5} \times \frac{{\cdot}0457}{36}\right)$$

$$= 17{\cdot}776 + 3{\cdot}141 = 20{\cdot}917 \text{ chains.}$$

(*c*) Long chords PQ, P′Q′ = X sec Ω

$$= L\left(1-\frac{1}{5}\frac{s}{R}\right)\sec 1°\ 40'$$

$$= 6{\cdot}283\left(1-\frac{3\times{\cdot}0457}{5\times 36}\right)\times 1{\cdot}0004 = 6{\cdot}281 \text{ chains.}$$

Φ the total spiral angle = 5° 00′.

3. Measure the total tangent distances IP, IP′, and drive pegs at the points P and P′. Set up the theodolite over the point-spiral P, and, sighting I, set off successive angles of 1′, 4′, 9′, etc., the chainmen laying down successive chord lengths *c* accordingly.

4. Set out from the tangent the whole deflection Ω, and along the direction thus given measure the long chord PQ. Drive a peg at Q. (Check by measuring the perpendicular from Q to the tangent, which distance should be 4*s*.)

5. Set up the theodolite over the *point-spiral-curve* Q, clamp the plates at 2Ω, and, sighting P, lay down a tangent at Q by bringing the plates to 0°. (Transit the telescope.) Proceed to run in the circular curve with 1-chain chords; if instructed, reducing the chainage of Q by adding the length L to the chainage of P. (See Note v.)

Chainage of Q = 76·32 + 6·283 = 82·603.

Tangential angle δ per chain $= \sin^{-1}\frac{1}{2R}$, or $\frac{1719}{36} =$ 47·75 minutes.

Hence, 1st tangential angle = ·397 × 47·75′ = 18·96′; 2nd ditto = 1·397 × 47·75 = 66·70′; and so on.

Length of circular portion $QQ' = \frac{\triangle - 2\Phi}{2\delta} = \frac{52°\ 30' - 10°\ 0'}{1°\ 35{\cdot}5'}$

= 26·806 chains.

Hence chainage of Q′ = 109·409; and chainage of P′ = 115·692.

6. Set up the theodolite over the *point-curve-spiral* Q′, and lay down a tangent thereat by setting off an angle $\frac{1}{2}(\Delta - 2\Phi)$ from Q. (Transit the telescope.) Lay off from this tangent an angle 2Ω, and observe if the cross hairs bisect the peg at the point-spiral-tangent P′. Set out from P′ successive deflections in minutes of the normal deflections from P′ plus n times the chord number N counted from P′, n being the total number of chords. (See Note vi.)

The foregoing process reduces to one step, which here consists in sighting Q with the plates clamped at $\frac{1}{2}(52° 30' - 10° 00')$ plus $2 \times 1° 40'$, or 24° 35′, turning the plates to 0°, and transiting the telescope, the cross hairs of which should bisect the peg at P′.

Incidentally, the back deflections from the long chord Q′P′ will be 171′, 144′, 119′, etc., n being 10 in $N(N+n)$, where N is the number of the point sighted as counted from P′.

7. *Interpolate pegs at even chainages either* (*a*) *with the theodolite or* (*b*) *by offsets from the chords.*

(*a*) Here the pegs may be sighted in with the theodolite at either P or Q of the first spiral, and either P′ or Q′ of the second spiral, the deflections from the tangents at these points being $N(3q+N)k$ for P and P′ and $N(3q - N)k$ for Q and Q′, while N is the distance in chains from q, the point occupied, which is 0 or L respectively.

Thus, if P and P′ be occupied, stations will occur on the 1st spiral PQ at ·68, 1·68, 2·68, 3·68, 4·68, and 5·68 chains from P, and on the 2nd spiral P′Q′ at ·69, 1·69, 2·69, 3·69, 4·69, and 5·69 chains from P′; the chordages will be 1·08

2·68, 4·27, 5·88, 7·45, and 9·04 on PQ, and 1·10, 2·69, 4·28, 5·88, 7·47, and 9·06 on P′Q′, and the corresponding deflections from the tangents at P and P′ will be respectively 1·17′, 7·18′, 18·23′, 34·34′, 55·50′, and 81·72′, and 1·21′, 7·24′, 18·32′, 34·57′, 55·80′, and 82·08′.

(*b*) Here the offsets z are calculated in inches by the formula $\frac{1}{4}''f(1-f)(3\mathrm{I}+1)c$, where c is the chord length in chains and I and f are respectively the integers and fractions in the chordages as reduced from chain distances from P and P′. Altogether this is the more satisfactory method.

The following are the offsets to the stations of the present example:

Station:	77	78	79	80	81	82	. . .
Offset:	$\frac{1}{16}''$	$\frac{1}{4}''$	$\frac{3}{8}''$	$\frac{5}{16}''$	$\frac{1}{8}''$	$\frac{3}{16}''$	. . .
Station:	110	111	112	113	114	115	
Offset:	$\frac{1}{4}''$	$\frac{7}{8}''$	$\frac{1}{4}''$	$\frac{3}{8}''$	$\frac{1}{4}''$	$\frac{1}{16}''$	

N.B.—When interpolating stations the chain should be carried round the spiral points already located.

The interpolation of stations is separated in order to emphasise the fact that this operation would normally be subsequent to tracing the spiral, while the latter operation itself might be deferred in the original location, the ruling points P, Q, and Q′ P′ temporarily sufficing.

Note v.—As will be seen in Problem 3, the circular curve may be set out with the theodolite at the shift tangent point A, neither Q nor Q′ being occupied. But even then a check on Q and Q′ is desirable.

Note vi.—As a rule it is better to insert the spirals from P and P′, particularly in the case of compound curves; though, on the other hand, construction in order through P, Q, . . . Q′, P′ is of practical appeal, and might be

adopted when the spirals are omitted from the primary location. (See Note vii. of Problem 1, and, particularly, the special note of Problem 4, where progressive alignment is extended to compound curves.)

OBSERVATIONS.

Problem 3.—The pegs a and a' are the tangent points of a circular curve of 30 chains radius, the pickets indicating the centre line of the straights.

It is required to replace the given curve by one of the same radius with transition curves, the speed being limited to 50 miles per hour, and the gradient of approach to 1 in 500.

Run in the shifted curve with the theodolite, inserting suitable transitions by offset measurements.

Procedure.—1. The magnitude of the intersection angle Δ not being known, set up the theodolite over the first tangent point a, and, backsighting along the prolongation of the straight, measure the angle to the second tangent point a', ascertaining thus the whole tangential angle, which is one-half the angle Δ. If a' is not visible, prolong both straights towards the point of intersection I, and, at convenient points x, y in the respective tangents, measure the angles Ixy, Iyx, these together being equal to the required angle Δ. When I is accessible, and is visible from both a and a', the angle Δ may be ascertained directly. (Fig. 6.)

For purposes of calculation, let the angle Δ be 48° 20′.

2. Ascertain the length of transition L for the limited speed V miles per hour, confining the gradient of approach to 1 in 500.

Here superelevation $E = \frac{V^2}{200R} = \frac{2500}{200 \times 30} = \cdot 417$ foot or 5″.

$L = 500E = 208 \cdot 5$ feet, which accords with the rule $\frac{V^3}{1381R}$, the latter giving L as 3·02 chains. Hence L = 3 chains conveniently.

3. Calculate the remaining elements and the data of the transition curves—viz., (*a*) the Shift; (*b*) the Tangent Excess *a*P; and (*c*) the offsets *y*, these preferably for tenths of the length L.

(*a*) Shift $s = \frac{L^2}{24R} = \frac{9 \times 12 \times 66}{24 \times 30} = 9{\cdot}9'' = {\cdot}0125$ chain.

(*b*) Tangent Excess $aP = s \tan \frac{1}{2}\Delta - \frac{1}{2}L\left(1 - \frac{1}{5}\frac{s}{R}\right)$

$= {\cdot}0125 \times {\cdot}4487 + 1{\cdot}5(1 - {\cdot}00008)$

${\cdot}0056 + 1{\cdot}5000 = 1{\cdot}506$ chains.

(*c*) Offsets *y*, preferably from the tangents consistently at successive distances of $\frac{1}{10}$ L, or 30 links. (See Note iii.)

Here the offsets will run as 1, 8, 27, 64, *125*, 216, 343, 512, 729, and 1,000 times y_1, where $y_1 = \frac{4s}{1000} = {\cdot}0396''$, x_1 being $\frac{1}{10}$L, and $y = \frac{4sx^3}{L^3}$ generally.

${\cdot}04''$ ${\cdot}32''$ $1{\cdot}07''$ $2{\cdot}53''$ *4·95″* $8{\cdot}55''$ $13{\cdot}58''$ $20{\cdot}28''$ $28{\cdot}87''$ $39{\cdot}6''$.

4. Measure the shift increment *a* to S, and erect a perpendicular SA inwards equal to the shift *s*. Drive a peg at the shift tangent point A. Measure with the theodolite at I the angle γ between S and A.

Measure along the tangent half the length of transition L from S to the point of spiral P, and the other half to the terminal point N. Erect at N the terminal ordinate NQ=4*s*, and drive a peg at the point of spiral and curve Q. Repeat the latter operations, determining the ruling points of the second spiral P′Q′. (See Note iv.)

5. Set up the theodolite over the point A, and lay

down the shift tangent thereat by turning off from I the angle γ, thus setting out a parallel by alternate angles. Sight along this shift tangent with the plates at 0°, and observe the tangential angle to Q. Measure the chord AQ, and note if its length accords with the tangential angle as given by $\sin^{-1}\frac{AQ}{2R}$. This checks the position of Q.

Run in the circular portion QQ′ from the same backsight, using 1-chain chords, and assuming Q to be of even chainage, unless otherwise instructed. Check the subchord closing on Q′ against its tangential angle in order to ascertain if Q′ is correctly on the circular curve. (See Note v.)

6. Insert the transition curves by offset measurements, in this case squaring such distances from the tangents, preferably at ten equal intervals.

Here the offsets will vary as the cubes of unit distances of 30 links from P and P′; thus—

P Q

0—$\frac{5}{16}''$ $1\frac{1}{16}''$ $2\frac{9}{16}''$ $4\frac{15}{16}''$ $8\frac{9}{16}''$ $13\frac{9}{16}''$ $20\frac{1}{4}''$ $28\frac{7}{8}''$ $39\frac{5}{8}''$.

Note iii.—When the shift is great or the radius is small, it is usually preferable to insert the spirals, half from the tangent PS, and half from the redundant portion AQ of the curve, the intervals being measured along the latter. Thus in the present case, the following offsets will occur on either side of the shift line AS:

0 at P and Q; $(\frac{1}{5})^3\frac{s}{2}$ at Pts. 1 and 9; $(\frac{2}{5})^3\frac{s}{2}$ at Pts. 2 and 8; $(\frac{3}{5})^3\frac{s}{2}$ at Pts. 3 and 7; $(\frac{4}{5})^3\frac{s}{2}$ at Pts. 4 and 8; $(\frac{5}{5})^3\frac{s}{2}$ at Pt. 5, on the shift line AS.

Note iv.—When the foregoing process supersedes tangent offsets the *entire* circular curve AA′ should be run in before the spirals. In this connection, the points on the redundant portions AQ, Q′A′ should be abreast the points on the spirals; otherwise, offsetting from estimated points on the curve will render the method considerably less accurate than that described in the text. However, the redundant portions may be run in with chords $\frac{1}{10}$L and corresponding tangential angles, 1, 2, 3, . . . 10 times 171·9 $\frac{L}{R}$ minutes. Or, if expedient, the portions may be constructed by the platelayer's method with chord offsets *o* corresponding to chords of $\frac{1}{10}$L chain, *o* being thus equal to $\frac{L^2}{100R}$. Of course, a template would avoid the use of either artifice.

Note v.—Though commonly negligible, the chainage of Q will be slightly greater than the chainage of P plus the length of transition L. Hence, in order to run in the pegs on the circular portion at correct chainages, it may be advisable to determine the chainage of Q by adding to the chainage of P half the length L plus the quotient $\frac{\beta}{\delta}$, where δ is the tangential angle per chain and β that for AQ, or $\sin^{-1}\frac{AQ}{2R}$. If, however, offsets from the redundant portion are employed, the chainage of Q will be the chainage of P plus the length L; and the chainage of A, the commencement of the circle, will be this sum less half the length L, the length of transition being twice that of the circular portion displaced.

Thus, if P is 33·60 chains and L 3 chains, the chainage of Q will be 36·60, and that of A this chainage less 1·5 chains, or 35·10. Hence if the first tangential angle from the shift tangent at A be set out as 0·90 δ, the pegs will be "chained through" on the curve. If, in addition, it is desired to

interpolate a peg at Q, the tangential angle to Q must be 1·5 δ, where δ, as before, is the tangential angle per chain.

Chainage pegs on the transitions can be inserted best by direct measurement round the points located by offsets.

OBSERVATIONS.

Problem 4.—The central angles of a compound with 4° and 4° 30′ branches scale and reduce respectively as 28° 44′ and 21° 23′. It is decided, however, to supersede this compound by one of the same respective radii with spirals, keeping the point of compound curvature as nearly as may be to the point fixed by the given central angles.

Effect the proposed change, assuming the working speed to be the maximum for the sharper branch.

PROCEDURE.—1. Set up the theodolite over the point of intersection I, and measure the angle supplementary to the intersection angle Δ.

For purposes of calculation let the angle Δ actually measure 50° 00′. But since the central angles α and β should together be equal to Δ, the scaled values must be amended; say, by subtracting 4′ from the larger and 3′ from the smaller angle. Thus, $\alpha = 28° 40'$; $\beta = 21° 20'$.

2. Calculate and measure the normal tangent lengths T and t, and establish temporary pegs at the corresponding tangent points a and a'. (*Also as a check, calculate the apex distance Ic, and establish a temporary peg at c. See data of 3.)

Thus:

$$T = \frac{R \text{ vers}\Delta - (R - r) \text{ vers } (\Delta - \alpha)}{\sin \Delta} \qquad (R = 1432{\cdot}7 \text{ feet.})$$

$$= \frac{1432{\cdot}7(1 - \cos 50°) - (1432{\cdot}7 - 1273{\cdot}6)(1 - \cos 21° 20')}{\sin 50}$$

$$= \frac{1432{\cdot}7 \times {\cdot}3572 - 159{\cdot}1 \times {\cdot}0685}{{\cdot}7660}$$

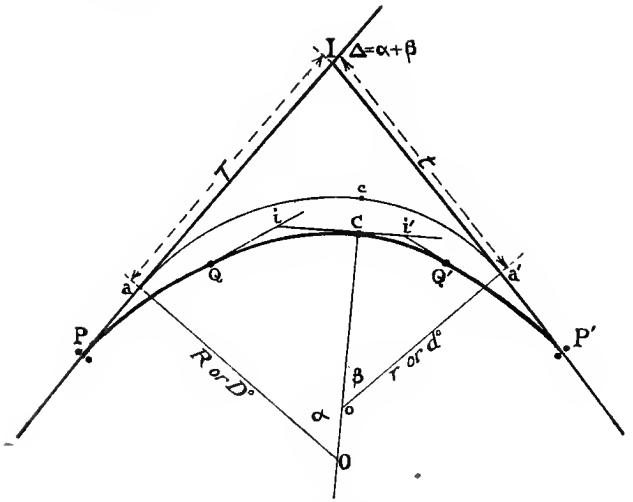
$$= 653{\cdot}9 \text{ feet.}$$

$$t = \frac{r \text{ vers}\triangle + (R - r) \text{ vers } (\triangle - \beta)}{\sin \triangle} \qquad (r = 1273{\cdot}6 \text{ feet.})$$

$$= \frac{1273{\cdot}6(1 - \cos 50°) + (1432{\cdot}7 - 1273{\cdot}6)(1 - \cos 28° 40')}{\sin 50°}$$

$$= \frac{1273{\cdot}6 \times {\cdot}3572 + 159{\cdot}1 \times {\cdot}1226}{{\cdot}7660}$$

$$= 619{\cdot}3 \text{ feet.}$$



$$\text{Vers.}\alpha = \frac{t \sin.\Delta - r \text{ vers.}\Delta}{R - r}$$

$$\text{Vers.}\beta = \frac{R \text{vers}\Delta - T \sin.\Delta}{R - r}$$

FIG. 7.

Also: $Ic^2 = T^2 + ac^2 - 2T \cdot ac \cdot \cos \frac{1}{2}a$; where $ac = 2R \sin \frac{1}{2}a$

$= 709{\cdot}5$ feet.

$Ic^2 = (653{\cdot}9)^2 + (709{\cdot}5)^2 - 2 \times 653{\cdot}9 \times 709{\cdot}5 \times {\cdot}9689$

$= 31{,}570$; and $Ic = 177{\cdot}7$ feet.

Sin $aIc = \frac{ac}{Ic} \sin \frac{1}{2}a = \frac{709{\cdot}5}{177{\cdot}7} \times {\cdot}2476 = {\cdot}9887$; and $aIc = 81° 23'$.

3. Assuming the conditions of time approach, the maximum speed of the sharper branch will obtain—viz., $100 \div \sqrt{d}$ miles per hour. Accordingly the length of transition l to this branch will be $600 \div \sqrt{d}$ feet, and s the normal value—viz., 2·618 feet. The length of transition L to the other branch will have to be found tentatively; say, as the above speed multiplied by the superelevation in inches. But since no central transition is to be considered, S sec α must equal s sec β; and if the corresponding lengths L and l are incompatible, the composite curve may have to be shifted bodily along the tangents.

Thus, S should equal $\dfrac{s \sec \beta}{\sec \alpha}$

$$= 2{\cdot}618 \times \frac{{\cdot}8774}{{\cdot}9315} = 2{\cdot}466 \text{ feet.}$$

Hence L should equal $\sqrt{24 \times \mathrm{RS}}$

$$= \sqrt{24 \times 1432{\cdot}7 \times 2{\cdot}466} = 291{\cdot}2 \text{ feet.}$$

But under normal circumstances L can be taken tentatively as $6V_m$, where V_m is the controlling speed. That is, $L = 6 \times 47{\cdot}1 = 282{\cdot}6$ feet, which also is the time approach length l.

Hence, shifting the composite curve is not to be considered, and L and l may be taken respectively as 291·2 feet and 282·6 feet with corresponding shifts S and s of 2·466 feet and 2·618 feet. (* Also, in the present case, it would be inadvisable to reconsider the entire construction, keeping the point of compound curvature at c instead of letting it move inwards S sec α along *co.*)

4. Calculate the remaining elements and the data of the spirals, assuming the latter to be traced with

chords C and c corresponding to deflections of 1, 4, 9, etc., minutes.

(a) Tangent excess $a\mathrm{P} = \mathrm{S}\tan\alpha + \frac{1}{2}\mathrm{L}$
$= 2{\cdot}47 \times {\cdot}5467 + 145{\cdot}6 = 147{\cdot}0$ feet.

,, ,, $a'\mathrm{P}' = s\tan\beta + \frac{1}{2}l$
$= 2{\cdot}62 \times {\cdot}3906 + 141{\cdot}3 = 142{\cdot}3$ feet.

Hence, total tangent distances, IP, IP'—

$\mathrm{T}_s = \mathrm{T} + a\mathrm{P} = 653{\cdot}9 + 147{\cdot}0 = 800{\cdot}9$ feet.

$t_s = t + a'\mathrm{P}' = 619{\cdot}3 + 142{\cdot}3 = 761{\cdot}6$ feet.

(b) Total deflection angles, Ω and ω,

$$\Omega = \frac{\mathrm{LD}}{10} = \frac{291{\cdot}2 \times 4}{10} = 1^\circ\ 56{\cdot}48',$$

$\omega = \sqrt{d} = 2^\circ\ 7{\cdot}26'$, being for time approach.

Whence $\Phi = 5^\circ\ 49{\cdot}44'$; $\phi = 6^\circ\ 21{\cdot}78'$.

Long chords PQ, P'Q', or X sec Ω and x sec ω respectively.

$$\mathrm{PQ} = \mathrm{L}\left(1 - \frac{3}{5}\frac{\mathrm{S}}{\mathrm{R}}\right)\sec\Omega = 290{\cdot}9 \times \sec 1^\circ\ 57' = 291{\cdot}1 \text{ feet.}$$

$$\mathrm{P'Q'} = l\left(1 - \frac{3}{5}\frac{s}{r}\right)\sec\omega = 282{\cdot}2 \times \sec 2^\circ\ 7' = 282{\cdot}4 \text{ feet.}$$

(c) Chord lengths per minute^2 deflection,

$$\mathrm{C} = \sqrt{\frac{10\mathrm{L}}{\mathrm{D}}} = \sqrt{\frac{10 \times 291{\cdot}2}{4}} = 26{\cdot}98 \text{ feet.}$$

$$c = \sqrt{\frac{10l}{d}} = \sqrt{\frac{10 \times 282{\cdot}6}{4{\cdot}5}} = 25{\cdot}06 \text{ feet.}$$

5. Measure the total tangent distances T_s, t_s, and drive three pegs at the points P and P'.

Set up the theodolite over the *point-spiral* P and sight along the tangent with the plates clamped at 0°; then set off successive deflections of 1', 4', 9', etc., the chainmen measuring the chord lengths accordingly. Sight with the whole deflection Ω and, measuring the

long chord, establish the *point-spiral-curve* Q. Note if the subchord closing on Q agrees with this angle, its length being the fraction of a chord expressed by the excess of $\sqrt{\Omega}$ over the number N of the preceding chord.

Thus: Ω being 116·5′, the preceding even square is 100′, and $\sqrt{116{\cdot}5} - \sqrt{100}$ gives a subchord of ·794C, or 21·42 feet.

6. Set up over the *point-spiral-curve* Q, and lay down a tangent thereat by sighting P with the plates clamped at 2Ω, and then bringing them to zero. Transit the telescope. [Sometimes it is convenient to establish a temporary intersection peg i by measuring along this tangent a distance of $R \tan \frac{1}{2}(\alpha - \Phi)$]. From this tangent run in the circular portion QC, setting off successive deflections of 2° per 100 feet chord. Locate the *P.C.C.*—C—by setting off the whole deflection angle $\frac{1}{2}(\alpha - \Phi)$ and measuring the chord QC equal to $2R \times \sin \frac{1}{2}(\alpha - \Phi)$. (If the latter measurement be impossible, line in two arrows p_1, p_2, one on each side of C as determined in the following step.) Observe if the subchord closing on C agrees with the difference between the last two deflection angles, $(\alpha - \Phi) \div D$ being the length of the portion QC in hundreds of feet.

Length of circular portion QC

$$= (28° 40′ - 5° 49{\cdot}5′) \div 4° = 571{\cdot}0 \text{ feet.}$$

7. Set up over the point of compound curvature C, and lay down a tangent thereat by sighting Q with the plates clamped at $\frac{1}{2}(\alpha - \Phi)$, and then bringing them to 0. (Observe if the cross wires bisect the temporary peg at i. Otherwise, obtain the correct position by shifting the instrument in the line of the arrows p_1 and p_2.) Transit the telescope. [Sometimes it is convenient to

establish a temporary intersection peg i' by measuring along this tangent a distance $r \tan \frac{1}{2}(\beta - \phi)$]. Proceed to run in the circular portion CQ′, setting off deflections of 2° 15′ per 100 feet chord. Locate the end Q′ of the curve CQ′ by sighting with the whole deflection angle $\frac{1}{2}(\beta - \phi)$ and measuring the whole chord CQ′ equal to $2r \sin \frac{1}{2}(\beta - \phi)$. (If the latter measurement be impossible, line in two arrows, q_1, q_2, one on each side of Q′ as determined in the following step. See Note vii.) Observe if the subchord closing on Q′ agrees with the difference between the last two deflection angles, $(\beta - \phi) \div d$ being the length of the portion CQ′ in hundreds of feet.

Length of circular portion CQ′

$$= (21° 20′ - 6° 21{\cdot}8′) \div 4° 30′ = 332{\cdot}7 \text{ feet.}$$

8. Set up over the *point-curve-spiral* Q′, and lay down a tangent thereat by sighting C with the plates clamped at $\frac{1}{2}(\beta - \phi)$ and then bringing them to 0°. (Observe if the cross wires bisect the temporary peg at i'.) Transit the telescope. From this tangent lay off twice the total deflection ω of the second spiral, and observe if the cross wires bisect the peg at the *point-spiral-tangent* P′. If such is not the case, first check the position of Q′ by measuring carefully the terminal ordinate Q′N′. (If error is evident, shift the instrument in the line of the arrows q_1, q_2 until the correct deflection gives P′ together with Q′N′ equal to $4s$. Here the temporary intersection peg i' serves in the case of obstructions in the whole chord CQ′.)

9. Set up the theodolite over the *point-spiral-tangent* P′, and lay off from the tangent successive

deflections of 121′, 100′, 81′, etc., the chainmen measuring first the subchord from Q′, and then successive chords of length c.

Here the first descending deflection will be 121′, the even square under ω, or 127′, and the corresponding subchord will be $(\sqrt{127}-\sqrt{121})c$ or $\cdot 292 \times 25\cdot06$ feet.

Interpolating Stations.—Stations at 100 feet intervals on the spirals may be inserted by the methods described in Problem 2, the chordages being the ratios of the station distances in feet from P and P′ to the chord lengths C and c, also in feet.

Thus, if P, the *P.S.*, is at $21+62$, Q, the *P.S.C.*, will be at $24+53\cdot2$, C, the *P.C.C.*, at $30+24\cdot2$, Q′, the *P.C.S.*, at $33+56\cdot9$, and P′, the *P.S.T.* at $36+39\cdot5$.

Hence stations on the spirals will occur at the following chordages:

On PQ:	$\frac{38\cdot0}{26\cdot98}$	$\frac{138\cdot0}{26\cdot98}$	$\frac{238\cdot0}{26\cdot98}$
On P′Q′:	$\frac{60\cdot5}{25\cdot06}$	$\frac{160\cdot5}{25\cdot06}$	$\frac{260\cdot5}{25\cdot06}$

the latter values appertaining to observations from P′.

N.B.—The occupation of the points P, Q, C, Q′, P′, in order is given on purpose to demonstrate the artifices by which tangency may be obtained at these ruling points in progressive construction. However, in compounding generally, it is usually more satisfactory to run in the first branch through P and the second branch through P′, the position of C being exactly located by intersections from Q and Q′. (See Problem 5.)

OBSERVATIONS.

Problem 5.—The pegs a, a', and c indicate respectively the tangent points and the point of compound curvature of a compound curve of 36 and 25 chains radii, the pickets indicating the centre line of the straights.

Replace the given compound by curves of the same respective radii, inserting suitable transition curves.

N.B.—Since the radii R and r differ by about 30 per cent. it may be necessary to consider a transition replacing the point of compound curvature.

PROCEDURE.—1. Set up the theodolite over the tangent point a, and measure the angle between the tangent and the point of compound curvature c. This will be one-half the central angle α of the first branch, the radius of which is R (Fig. 7). Set up over the point of compound curvature c, and measure the interior angle between a and a'. This will be the supplement of half the sum of the central angles α and β.

For purposes of calculation, let α and β be 36° 52′ and 20° 40′ respectively, the former subtending the 36 chain curve, the latter the 25 chain curve.

2. Investigate the compound as to suitable transitions, basing the work on the approaches from the straights and the maximum speed of the sharper branch.

(*a*) Lengths of transition, L, l_c, and l. Assuming the conditions of time approach, the length l of the sharper curve will be $\sqrt{r}$, or 5 chains, while that L of the flatter one will be given by $\frac{V^3}{1381R}$ as 3·346, V being 55 m.p.h. in accordance with the rule $11\sqrt{r}$.

Accordingly the end shifts S and s will be 10·22″ and 33·0″ respectively. These will produce central displace-

ments of S sec α and s sec β, and the difference of these will be the gap d of the central transition. Thus $d = \frac{33''}{\cdot 9357} - \frac{10 \cdot 22''}{\cdot 8000} = 22 \cdot 5$ inches. Now the shift s_c in accordance with the formula $s_c = \frac{1}{24}\left(1 - \frac{r}{R}\right)^3$ would only require to be $0 \cdot 94''$, and the length l_c accordingly 1·494 chains by the formula $l\left(1 - \frac{r}{R}\right)$. This result accords with the value of l_c given by multiplying the difference in superelevation ($0 \cdot 5' - \cdot 347'$) by 660, which is the gradient of approach of the sharper curve—*i.e.*, $l_c = 1 \cdot 53$ chains. On the other hand, a shift under $3''$ could not be conveniently negotiated at the point of compound curvature.

There are three ways of considering the problem, excepting that of dispensing with the central transition, as doubtless would be done in the case in question.

1. Assuming the shift s_c at a minimum of $3''$, say, finding $l_c = 2 \cdot 727$ chains, and finally shifting the sharper curve along the tangent, thus approximating the original location as nearly as may be.

2. Inserting an absurdly long transition l_c, which for the gap of 22½ inches would be 7·47 chains, giving a gradient of approach of 1 in 3220.

3. Lengthening the end transition to the flatter curve such that a convenient gap of, say, 4 inches would result. But this would mean both inordinately long central and end transitions.

Hence it is evident that the conditions that demand central transitions result normally in a large gap, while such required by radii differing by 50 per cent. are only about 4 inches as far as difference in curvature is concerned. However, other considerations may warrant a large gap even though the transition l_c is consequently excessive.

In order to introduce shifting a half of the composite

curve, let the first assumption be followed, a gap of 3″ being introduced. Then the shift S for the larger radius R should be such that $S = (s \sec \beta - 3'') \cos \alpha$, or (as in Fig. 4) AA′ should be (35·27 − 3) cos 36° 52′, or 17·99 inches. Hence, since AA′ is greater than S, the point B must be slid along the tangent a distance Bb′, which is expressed by

$$\frac{(s \sec \beta - 3) \cos \alpha - S}{\sin (\alpha - \beta)} = \frac{17{\cdot}99 - 10{\cdot}22}{{\cdot}8437} = 9{\cdot}21''.$$

4. Calculate the remaining elements and data of the transition curves, deciding to spiral through with half-chain chords and to insert the central transition with offsets.

(*a*) Total end deflections Ω and ω, and long chords PQ, P′Q′:

$$\Omega = 573 \frac{3{\cdot}346}{36} = 53{\cdot}3'; \text{ and } \Phi = 2° 39{\cdot}8'.$$

$$\omega = 573 \frac{5{\cdot}00}{25} = 1° 54{\cdot}6'; \text{ and } \phi = 5° 43{\cdot}8'.$$

$$PQ = X \sec \Omega = L\left(1 - \frac{3}{5}\frac{S}{R}\right) \sec 53'$$

$$= 3{\cdot}345 \times 1{\cdot}00012 = 3{\cdot}345 \text{ chains.}$$

$$P'Q' = x \sec \omega = l\left(1 - \frac{3}{5}\frac{s}{r}\right) \sec 1° 55'$$

$$= 4{\cdot}995 \times 1{\cdot}00058 = 4{\cdot}998 \text{ chains.}$$

(*b*) Shift increment for PQ

$= S \tan \alpha = 10{\cdot}22 \times {\cdot}7499 = 7{\cdot}66''$, or ·01 chain.

Shift increment for P′Q′

$= s \tan \beta = 33 \times {\cdot}3772 = 12{\cdot}45''$.

But this will be reduced by 9·21″, and will thus be equal to $3\frac{1}{4}''$, or ·004 chain.

(*c*) Spiral extension $SP = \frac{1}{2}L\left(1 - \frac{1}{5}\frac{S}{R}\right) = 1{\cdot}673$ chains.

Spiral extension $S'P' = \frac{1}{2}l\left(1 - \frac{1}{5}\frac{s}{r}\right) = 2{\cdot}499$ chains.

5. Set up the theodolite over the *point-spiral* P, and locate Q by the long chord PQ and the total angle Ω. Run in the transition PQ, using half-chain chords in the manner described in Problem 1. Set up over Q, and run in the circular portion QC_1 of radius R. Observe if the end of this curve is at S sec α inwards from the point of compound curvature c.

6. Set up over the *point-spiral-tangent* P′, and locate Q′ by the long chord P′Q′ and the total deflection ω. If the chainage of Q′ has been computed, the spiral may be run in from this station. Set up over Q′, and run in the circular portion C_2 Q′ of radius r, setting off from the tangent at Q′ the differences between $\frac{1}{2}(\beta - \phi)$ and the tangential angles for an instrument at C_2.

7. Run in the central portion RR′ by offsets from the displaced circular portions RC_1, C_2R', varying the ordinates as the cubes of the distances from the *point-spiral-compound* C, where the transition bisects the shift. (Fig. 4)

Thus with five intervals of 14·9 links on each side of C, the offsets from R and R′ will be respectively:

0; $(\frac{1}{5})^3\ 1\frac{1}{2}''$; $(\frac{2}{5})^3\ 1\frac{1}{2}''$; $(\frac{3}{5})^3\ 1\frac{1}{2}''$; $(\frac{4}{5})^3\ 1\frac{1}{2}''$; and $1\frac{1}{2}''$.

Incidentally, the chainages of the compound from Q to Q′ will be the same as though no transition were inserted.

OBSERVATIONS.

Problem 6.—The pegs a and a' are the tangent points of a circular curve of 30 chains radius, the pickets indicating the centre line of the straights. It is required to replace the given curve by one provided with transition curves, the greater portion of the curve remaining undisturbed.

PROCEDURE.—1. Assume the conditions of time approach, making s thus constant and equal to 33 inches.

Also assume a slightly smaller radius, 29 chains, say, for the amended radius R_a. Then γ, the angle cut out of each half of the given curve, will be such that vers $\gamma = \frac{s}{R_o - R_a}$, R_o being the radius of the given curve in chains.

Thus: Vers $\gamma = \frac{1}{24}\left(\frac{1}{30-29}\right) = \frac{1}{24} = \cdot 041666$;

whence $\gamma = 16° 36'$.

Had $\triangle$ been less than 4γ, it would have been advisable to have assumed a larger value for R_a.

For purposes of calculation, let the intersection angle $\triangle$ be 67° 22′. Hence the undisturbed portion CC′ subtends a central angle of 35° 10′, and is equal in length to 35° 10 ÷ 1° 54·6′, or 18·325 chains, 57·3′ being the tangential angle δ per chain.

2. Calculate the elements and the data of the transition curves—viz., (*a*) Whole chord, aC or a'C′; (*b*) Tangent excess, aP or a'P′; (*c*) Total deflection, Ω, and long chord, PQ or P′Q′; and (*d*) Spiral deflection k per chain^2 or per chord c^2.

(*a*) Whole chords, aC, $C'a' = 2R_o \sin \frac{1}{2}\gamma$

$= 2 \times 30 \times \sin 8° 18' = 8 \cdot 662$ chains.

(*b*) Tangent excesses, Pa, a'P′

$= X - R_a \sin \Phi - (R_o - R_a) \sin \gamma$,

where X is the terminal abscissa and $\Phi = 3\Omega$, the total spiral angle.

Here $X - R_a \sin \Phi$ is practically half the length of transition L, which by the $\sqrt{R}$ rule is $\frac{1}{2}\sqrt{29}$, or 2·693 chains.

Thus, Pa or $P'a' = \frac{1}{2}\sqrt{29} - \sin 16° 36'$

$= 2{\cdot}693 - {\cdot}2857 = 2{\cdot}407$ chains.

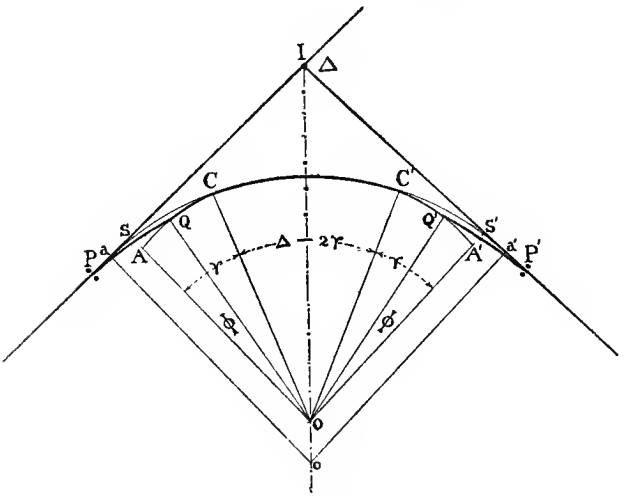


FIG. 8.

(*c*) Total deflection $\Omega = \dfrac{573}{\sqrt{R_a}}$

$= 106{\cdot}4'$; and $\Phi = 5° 19{\cdot}2'$.

Long chords PQ, $P'Q' = X \sec \Omega$

$= L\left(1 - \frac{3}{5}\frac{s}{R_a}\right) \sec 1° 46{\cdot}4'$

$= 5{\cdot}380 \times 1{\cdot}0005 = 5{\cdot}383$ chains.

(*d*) Deflection k per chain^2 = 3·669 minutes.

Deflection per chord^2, c = ·522 chain.

It follows that each circular portion QC or C′Q′ will be equal in length to $\gamma - \Phi \div 2\delta$ chains, where δ is the tangential angle for the sharpened curve. Thus, QC and C′Q′= 6·079 chains.

3. Measure the tangent excesses aP and a'P′, and establish with pegs the points P and P′. Set up over P, and run in the spiral PQ. Establish the point Q with the long chord and total deflection Ω.

4. Set up over Q and establish a tangent thereat by setting off from P an angle 2Ω. From this tangent run in the circular portion QC, if possible, establishing the point C with the chord QC, which is equal to $2R_a \sin \frac{1}{2}(\gamma - \Phi)$.

5. Set up over C, and lay down a tangent thereat by setting off from Q an angle $\frac{1}{2}(\gamma - \Phi)$. Observe if the angle between this tangent and a is equal to $\frac{1}{2}\gamma$. Check the circular portion CC′, if possible, establishing C′ with the whole chord $2R_o \sin \frac{1}{2}(\Delta - 2\gamma)$.

6. Set up over C′, and lay down a tangent thereat by setting off from C an angle equal to $\frac{1}{2}(\Delta - 2\gamma)$. Verify this position of C′ by observing if the angle to the given tangent point a' is equal to $\frac{1}{2}\gamma$. Before adjusting, measure the whole chord C′a' if possible. (See Note vi.)

7. From the tangent at C′ run in the circular portion C′Q′, measuring the whole chord C′Q′ if possible; or, failing this, adopt P′ instead of Q′ as the station in constructing the spiral Q′P′.

Note vi.—It is evident that there are various modes of procedure in constructing the composite curves, and, in consequence, the given routine may have to be amended considerably in order to conform with local conditions.

In general, it is best to locate the points of compound curvature C and C′ from either a and a' or Q and Q′, the latter as located from P and P′ respectively.

N.B.—It is well to note that when it is impossible to measure either of the whole chords C′Q′ and C′a', the point C′ may be fixed by the intersections from Q′ and a', or from C and a'.

OBSERVATIONS.

Problem 7.—The points a, C, and a' have been located respectively as the points of *curvature*, *compound curvature*, and *tangency* of a compound of 6° and 7°, the pickets indicating the centre line of the straights.

Replace the given compound by one provided with suitable spirals at the beginning and end, leaving the point of compound curvature undisturbed.

State the difference in length between the original and amended routes, the chainage of the tangent point a being recorded as 61+27 and the central angles of the 6° and 7° portions as 24° 16′ and 20° 40′ respectively.

PROCEDURE.—1. Set up the theodolite at the point of compound curvature C, and measure the exterior angle aCa', which should equal the sum of the central angles α and β. Set up over the tangent point a, and, backsighting along the tangent, measure the angle IaC, which should be equal to α.

For purposes of calculation, let the angles α and β check to their respective values of 24° 16′ and 20° 40′.

Since the central angles are small, it may be expedient merely to sharpen the radii R_o and r_o to R_a and r_a respectively, instead of compounding at intermediate points R and R′, and thus retaining a very considerable portion of the original curves (Fig. 9). On the former basis, calculate the reduced radii R_a and r_a corresponding to degrees D_a and d_a sufficiently increased to give the necessary space shifts S and s. (See Note ii.)

Thus: $\text{vers } \alpha = \dfrac{S}{(R_o - R_a)} : \text{vers } \beta \dfrac{s}{(r_o - r_a)}$.

2. Assuming the conditions of time approach, the shift s of the curve Q′C will have its normal value of 2·618 feet

in accordance with the length $600 \div \sqrt{d}$ feet, while the controlling speed of this sharper branch will determine L and consequently S of the approach to the flatter curve CQ. But, since no central transition is involved and the maximum speeds will not differ materially, the 6° branch may

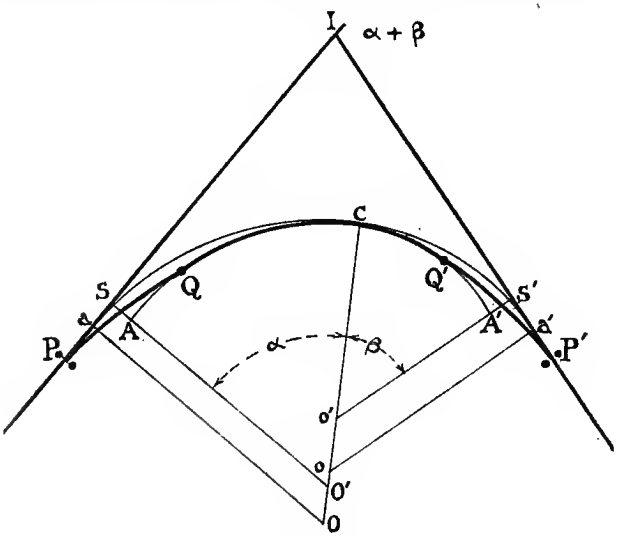


FIG. 9.

be elevated by transition for its own speed, and, thus introducing the normal shift s, will avoid the assuming of a value for R_a in determining S.

Hence vers 24° 16′

$$= \frac{2{\cdot}618}{(955{\cdot}37 - R_a)}; \text{ and } R_a = 925{\cdot}74 \text{ feet tentatively;}$$

vers 20° 40′

$$= \frac{2{\cdot}618}{(819{\cdot}02 - r_a)}; \text{ and } r_a = 778{\cdot}34 \text{ feet tentatively.}$$

Since these values respectively exceed 924·58 feet and 778·31 feet, the radii for 6° 12′ and 7° 22′, it will be advisable to increase the shifts to 2·721 and 2·620 by multiplying 2·618 by $\frac{30\cdot79}{29\cdot63}$ and $\frac{40\cdot71}{40\cdot68}$. Accordingly the lengths of transition L and l will be—

$$L = \sqrt{24\ R_a S} = \sqrt{24 \times 924\cdot58 \times 2\cdot721} = 245\cdot7 \text{ feet.}$$

$$l = \sqrt{24\ r_a s} = \sqrt{24 \times 778\cdot31 \times 2\cdot619} = 221\cdot2 \text{ feet,}$$

6° 12′ and 7° 22′ being adopted as the degrees D_a and d_a.

3. Calculate the remaining elements and data of the spirals, assuming the latter to be "run through," with $33\frac{1}{3}$ feet chords preferably.

(*a*) Tangent excesses, P*a* and P′*a*′; (*b*) Total deflections, Ω and *ω*, and long chords, PQ and P′Q′; and (*c*) Deflections, *k* per 100 feet[2].

(*a*) Tangent excesses:

$$aP = (X - R_a \sin \Phi) - (R_o - R_a) \sin a$$

$$= \tfrac{1}{2}L\left(1 - \frac{1}{5}\frac{S}{R_a}\right) - (R_o - R_a) \sin 24° 16'$$

$$= 122\cdot8 - 30\cdot79 \times \cdot41098 = 110\cdot1 \text{ feet.}$$

$$a'P' = \tfrac{1}{2}l\left(1 - \frac{1}{5}\frac{s}{r_a}\right) - (r_o - r_a) \sin 20° 40'$$

$$= 110\cdot5 - 40\cdot71 \times \cdot35293 = 96\cdot1 \text{ feet.}$$

(*b*) Total deflection $\Omega = \frac{573L}{R_a} = 2° 32\cdot27'$.

Total deflection $w = \frac{573l}{r_a} = 2° 42\cdot85'$.

$\Phi = 7° 36\cdot81'$; and $\phi = 8° 8\cdot55'$.

Long chord $PQ = L\left(1 - \frac{3}{5}\frac{S}{R_a}\right) \sec 2° 32'$

$= 245\cdot3 \times 1\cdot00098 = 245\cdot5$ feet.

$$\text{Long chord } P'Q' = l\left(l - \frac{3}{5}\frac{s}{r_a}\right) \sec 2^\circ 43'$$

$$= 220{\cdot}8 \times 1{\cdot}00112 = 221{\cdot}0 \text{ feet.}$$

(*c*) Deflections per $33\frac{1}{3}$ ft^2

$$= \left(\frac{1}{3}\right)^2 \frac{10D}{L} = \frac{25{\cdot}234}{9} = 2{\cdot}804 \text{ minutes}$$

$$= \left(\frac{1}{3}\right)^2 \frac{10d}{l} = \frac{33{\cdot}301}{9} = 3{\cdot}700 \text{ minutes.}$$

Assuming P to be 27′ rear of 61+27—viz., at 61+00—Q will be at 63+45·7, C at 66+14·3, Q′ at 67+84·3, and P′ at 70+5·5.

Hence the stations from P will be at 1·00, 2·00, or chordages 3·0 and 6·0, while intermediate points will occur at chordages 1, 2, . . . 4, 5, . . . and 7. Here the deflections will be merely $(1)^2$, $(2)^2$, $(3)^2$, etc., times 2·804 minutes.

Similarly the stations from P′ will be at ·945 and 1·945 or chordages 2·835 and 5·835, while intermediate points will occur at chordages ·835, 1·835 . . . 3·835, 4·835 . . . Here the deflections will be in order, $(\cdot 835)^2$, $(1{\cdot}835)^2$, $(2{\cdot}835)^2$, etc., times 3·700 minutes.

4. Measure the tangent excesses aP, a'P′, and establish the points of spiral P and P′. Set up the theodolite over the *point-spiral* P, and, backsighting along the tangent PI, set out the spiral PQ′, running through with $33\frac{1}{3}$ feet chords. Set out the whole deflection Ω, and, measuring the long chord PQ, establish the *point-spiral-curve* Q.

5. Set up over Q, and lay down a tangent thereat by turning off from P an angle 2Ω. (Transit the telescope). Run in the circular branch QC from this tangent, chaining through with 100 feet chords and

setting out deflections δ accordingly. Note if the subchord closing on C accords with the difference between the last two deflections.

6. Set up over the *point-compound-curvature* C, and lay down a tangent thereat by turning off from Q an angle equal to $\frac{1}{2}(a - \Phi)$. (Transit the telescope.) Run in the circular branch CQ′ from this tangent, chaining through with 100 feet chords and deflections δ' accordingly. Set out the whole deflection angle to Q′ as $\frac{1}{2}(\beta - \phi)$, fixing Q′ with the corresponding subchord. Check the position of Q′ by measuring the terminal offset N′Q′, which should equal $4s$. (See Notes appended to Problems 1, 4, and 6.)

7. Set up the theodolite over the *point-tangent* P′, and, backsighting along the tangent P′I, lay off the total deflection ω, checking thus the point Q′. Run in the second spiral Q′P′, setting off the tabulated deflections in descending order, the subchord coming first.

8. Calculate the length of the original route between the points P and P′, first finding the length a to a' by dividing the central angles by their respective degrees D° and $d°$, and adding the tangent excesses aP and a'P′. Compare this distance with the amended length already determined in the notes:

Original length a to $a' = 24° 16' \div 6° + 20° 40' \div 7°$

$= 404{\cdot}4 + 295{\cdot}2 = 699{\cdot}6$

Tangent excesses $= 110{\cdot}1 + 96{\cdot}1 = 206{\cdot}2$

Total distance a to a' $= 905{\cdot}8$ feet.

Amended distance from differences P′ and P = 905·5 feet.

Note ii.—When the lengths of the branches aC, C$'a'$ are considerable, it is usual to compound the sharpened curves with a considerable portion of the original curve, thus introducing angles Υ and v in the expressions. Here Υ and v must be assumed, as a rule, such that they are as small as possible consistent with the allowable difference in the degrees of curvature.

OBSERVATIONS.

Problem 8.—The pegs a, r, and a' indicate respectively the *points* of *curvature*, *reverse curvature*, and *tangency* of a true reverse with branches each of 4°. It is required to replace this curve with one having suitable spirals at the straights and at the reverse.

Give the ruling points of the most efficient improvement, the maximum deviation from the existing route being limited to 10 feet.

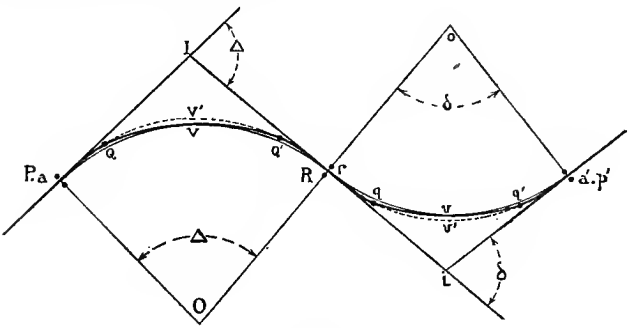


FIG. 10.

There are two ways of dealing with this problem, each making r the common point of spiral of two separate curves, the lengths of transition at the straight being equal to that at the reverse. Of these the first (A) involves sharpening the radius of each branch, keeping the vertex undisturbed, while the second (B) allows vertex displacement as well as reduction of radius.

For purposes of calculation, let the recorded values of the central angles α and β be assumed—viz., 52° 10′ and 56° 20′ respectively.

(A) Vertex Unchanged. Here $h=0$ in Eq. 3, so that $X - R_a \sin \Phi = (R_o - R_a) \sin \frac{1}{2}\Delta$. Then, time ap-

proach being assumed, R_a is determinate from the equation—

$$R_a + \frac{4\sqrt{R_a}}{\sin \frac{1}{2}\Delta} - R_o = 0.$$

Solving the quadratic:

$$\begin{aligned}\sqrt{R_a} &= -\frac{2}{\sin \frac{1}{2}\Delta} \pm \sqrt{R_a + \frac{4}{\sin^2 \frac{1}{2}\Delta}} \\ &= -\frac{2}{\sin 26^\circ\ 5'} \pm \sqrt{1432{\cdot}7 + \frac{4}{\sin^2 26^\circ\ 5'}} \\ &= -\frac{2}{{\cdot}4397} \pm \sqrt{1432{\cdot}7 + \frac{4}{{\cdot}1933}} \\ &= -4{\cdot}548 \pm \sqrt{1432{\cdot}7 + 20{\cdot}69} \\ &= -33{\cdot}576 \text{ feet; and } R_a = 1127{\cdot}4 \text{ feet.}\end{aligned}$$

R_a thus corresponds to D_a 5° 5′, the radius of which is 1127·5 feet.

(B) Vertex Displaced. On the same assumptions, a central displacement h being admissible, R_a is determinate from the equation—

$$R_a + 4\frac{\sqrt{R_a}}{\tan \frac{1}{2}\Delta} - (R_o - s) = 0.$$

Solving the quadratic:

$$\begin{aligned}\sqrt{R_a} &= -\frac{2}{\tan \frac{1}{2}\Delta} \pm \sqrt{(R_o - s) + \frac{4}{\tan^2 \frac{1}{2}\Delta}} \\ &= -\frac{2}{\tan 26^\circ\ 5'} \pm \sqrt{(1432{\cdot}7 - 2{\cdot}67) + \frac{4}{\tan^2 26^\circ\ 5'}} \\ &= -\frac{2}{{\cdot}4895} \pm \sqrt{1430 + \frac{4}{{\cdot}2396}} \\ &= -4{\cdot}0856 \pm \sqrt{1430 + 16{\cdot}69} \\ &= 33{\cdot}95 \text{ feet; and } R_a = 1152{\cdot}6 \text{ feet.}\end{aligned}$$

R_a is thus nearest to D_a 4° 58′, the radius of which is 1154·0 feet.

The consequent vertex displacement—

$$h = \frac{(R_c - R_a) \text{ vers } \frac{1}{2}\triangle - S}{\cos \frac{1}{2}\triangle}$$

$$= \frac{(1432{\cdot}7 - 1152{\cdot}6) \text{ vers } 26^\circ\ 5' - 2{\cdot}67}{\cos\ 26^\circ\ 5'}$$

$$= 28{\cdot}78 \text{ feet.}$$

Hence the process (B) is inadmissible, unless the length of transition is reduced, admitting a larger value of R_a and so reducing h. Therefore let it be assumed that the 5° 5′ curve may supersede the first branch, and on similar lines determine the amended degree d_a of the second branch. But since the angle $\triangle$ is less than δ, the radius R_a will be less than r_a, and, in consequence, the time approach length $8\sqrt{R_a}$ feet may be adopted for the four transitions.

Thus L, L′, l', and $l = 268{\cdot}6$ feet, as against 266·2 feet by the rule $600 \div \sqrt{D_a}$.

Calculate the remaining elements of the spirals:

(*a*) Total deflections, $\Omega = 573\ \frac{L}{R_a}$; $\omega = 573\ \frac{L}{r_a}$.

(*b*) Long chords, aQ, $Q'r = X \sec \Omega = L\left(1 - \frac{3}{5}\frac{S}{R_a}\right) \sec \Omega$.

Long chords, rq, $q'a' = x \sec \omega = L\left(1 - \frac{3}{5}\frac{s}{r_a}\right) \sec \omega$.

(*c*) Reduced central angles, $(\triangle - 2\Phi)$ and $(\delta - 2\phi)$, Φ and ϕ being 3Ω and 3ω respectively.

1. Theodolite at Point of Curvature a. Measure along a direction Ω to the tangent the long chord aQ, establishing the *point-spiral-curve* Q.

2. Theodolite at Point of Reversed Curvature r. Measure along a direction Ω to the back tangent the long chord rQ', establishing the *point-curve-spiral* Q′. Measure along a direction ω to the forward tangent

the long chord *rq*, establishing the *point-spiral-curve q*.

3. Theodolite at Point of Tangency *a'*. Measure along a direction ω to the back tangent the long chord *a'q'*, establishing the *point-curve-spiral q'*.

4. Check the construction by measuring the terminal ordinates QN and Q'N' and *qn*, *q'n'*, which should be equal to 4S and 4*s* respectively. If possible, set up over Q and *q*, and check the alignment of Q' and *q'* by setting off from tangents at the points occupied angles equal to $\frac{1}{2}(\Delta - 2\Phi)$ and $\frac{1}{2}(\delta - 2\phi)$ respectively.

Note.—Cases of improvement of existing lines will seldom admit of considerable reduction of radii or large vertex displacements; and, in consequence, if *h* is not fixed, a shorter transition than $a\sqrt{R_a}$ will have to be assumed, $\frac{2}{3}a\sqrt{R_a}$, for instance.

OBSERVATIONS.

Problem 9.—The pegs a, v, and a' indicate respectively the *point* of *curve*, the *vertex*, and the *point* of *tangent* of the given 28 chain curve, the centre line of the straights being reproduced with pickets.

It is required to replace this curve by one with transitions that approximates the present length of route, and is at the same time within 2 feet of the present alignment.

Set out the required improvement, deferring the insertion of transition curves, and state the maximum difference between the lengths of the original and amended routes.

PROCEDURE.—1. Set up the theodolite over the tangent points a and a', and, backsighting along the whole chord or the straights, measure the angle to the vertex v. In either case this angle should be $\frac{1}{4}\Delta$, Δ being the intersection angle. But in effecting improvements, it is often impossible to verify satisfactorily the intersection angle, and, in consequence, this angle is to be taken at the value ascertained in the original location.

Hence, for purposes of calculation, let the angle Δ have its recorded value of 42° 36′.

2. Assuming the conditions of time approach, so that $s = \cdot 0417$ chain, determine R_a tentatively from the relation $h \cos \frac{1}{2}\Delta = (R_o - R_a) \text{ vers } \frac{1}{2}\Delta - s$, where h is the vertex displacement on amending the radius from R_o to R_a. (See Note ii.)

Hence, on taking h as ·0300 chain,

$$\cdot 0300 \times \cos 21° 18' = (28 - R_a) \text{ vers } 21° 18' - \cdot 0417;$$
$$\cdot 0300 \times \cdot 9317 = (28 - R_a) \cdot 06831 - \cdot 0417.$$

Whence $28 - R_a = 1 \cdot 02$ chains.

Let Ra be taken finally as 27 chains, thus reducing h to about 22½ inches.

Then L, the length of transition, will be $\sqrt{R_a}$, or 5·196 chains.

3. Calculate the required elements of the spirals and curve—viz., (*a*) the Tangent excess, aP or a'P'; (*b*) the Total deflection, Ω and the long chord, PQ or P'Q'; (*c*) the Whole tangential angle, Θ; (*d*) the Tangential angle, δ per chain; and (*e*) the Length of circular curve. Also compute the difference in length between the original and amended routes.

(*a*) Tangent excess $aP = \frac{1}{2}L\left(1 - \frac{1}{5}\frac{s}{R_a}\right)$

$= 2{\cdot}598\left(1 - \frac{1}{5 \times 24 \times 27}\right)$

$= 2{\cdot}597$ chains.

(*b*) Total deflection $\Omega = \frac{573L}{R_a} = \frac{573}{5{\cdot}196} = 1° 50{\cdot}28'$; whence $\Phi = 5° 30{\cdot}84'$.

Long chord $PQ = L\left(1 - \frac{3}{5}\frac{s}{R_a}\right) \sec \Omega$

$= 5{\cdot}196(1 - \frac{3}{5}\cdot\frac{1}{24}\cdot\frac{1}{27}) \sec 1° 50{\cdot}28'$

$= 5{\cdot}191 \times 1{\cdot}0005 = 5{\cdot}194$ chains.

(*c*) Whole tangential angle $\Theta = \frac{1}{2}(\triangle - 2\Phi)$

$= \frac{1}{2}(42° 36' - 11° 1{\cdot}7')$

$= 15° 47'$.

(*d*) Tangential angle δ per chain $= \frac{1719}{R_a} = \frac{1719}{27} = 1° 3{\cdot}7'$.

(*e*) Length of Original Half-Route

$= \frac{1}{2}L\left(1 - \frac{1}{5}\frac{s}{R_a}\right) - (R - R_a)\tan\frac{1}{2}\triangle + s\tan\frac{1}{2}\triangle + R_a\frac{1}{2}\triangle$

$= 2{\cdot}597 - {\cdot}3899 + {\cdot}0417 \times {\cdot}3899 + 28 \times {\cdot}3718$

$= 12{\cdot}633$ chains.

Length of Amended Half-Route

$= R_a(\frac{1}{2}\Delta + \Phi) = 27(\cdot 3718 + \cdot 09623) = 12\cdot 636$ chains.

Hence difference in length is ·6 link, or $4\frac{3}{4}$ inches. (See Note iii.)

4. Measure from the tangent points a and a' the equal tangent excesses aP and a'P′, and establish with pegs the points P and P′.

5. Set up over the *point-spiral* P, and, backsighting along the tangent, set out the total deflection Ω. Measure along the direction thus given the long chord PQ, and establish the point Q.

6. Set up over the *point-spiral-curve* Q with the plates clamped at 2Ω, backsight on P, bring the plates to 0°, and transit the telescope. From the tangent thus established at Q, set out the circular portion QQ′ with 1 chain chords, reducing the chainage of Q if "chaining through" is stipulated. During progress interpolate a peg at the amended vertex V, setting out a tangential angle of $\frac{1}{4}(\Delta - 2\Phi)$ to the corresponding chainage around the curve. Verify the position of Q′, the *point-spiral-curve.*

7. Set up over the point Q′, and check the position of the *point-spiral-tangent* P′ by backsighting on Q with the plates set at $\frac{1}{2}(\Delta - 2\Phi)$, transiting, and then bringing the vernier past 0° to 2Ω.

Note ii.—The mutual relations between h, s, and R_a may be opportunely reviewed at this juncture.

(*a*) When $h = 0$, $s = (R_o - R_a)$ vers $\frac{1}{2}\Delta$.

(*b*) When $R_o = R_a$, $s = -h \cos \frac{1}{2}\Delta$, the displacement being inwards.

(*c*) When $R_o = R_a + s$, $s = -h$.

(*d*) When $s \tan \frac{1}{2}\Delta = (R_o - R_a)(\tan \frac{1}{2}\Delta - \frac{1}{2}\Delta)$ in the

matter of equal routes, $h \cos \frac{1}{2}\triangle = \frac{1}{2}s$, and $s = \frac{2}{3}(R_o - R_a) \times$ vers $\frac{1}{2}\triangle$.

The relation $s = 2\ h \cos \frac{1}{2}\triangle$ is derived by expanding tan $\frac{1}{2}\triangle$ and sin $\frac{1}{2}\triangle$ after substituting $s \tan \frac{1}{2}\triangle \div (\tan \frac{1}{2}\triangle - \frac{1}{2}\triangle)$ from the equation of equal routes in the general expression $h \cos \frac{1}{2}\triangle = (R_o - R_a)$ vers $\frac{1}{2}\triangle - s$. The derived results of (*d*) thus enable the engineer to deal directly with the case of compound curves. (See p. 95.)

Note iii.—It follows from paragraph (*d*) of the preceding note that the assumption of equal routes would have led to a displacement h under 18 inches, and, in general, the construction would have more closely approximated the original alignment. But since R_a would have been taken as 27 chains, the shift s would be necessarily increased to 3 feet, in order to yield the conditions desired. Merely by chance the improvement in the text is to be preferred.

OBSERVATIONS.

Problem 10.—The peg *a* is the *point* of *curvature* of a 3° curve, the peg *c* the *point* of *compound* with a 3° 30′ curve, and the peg *a′* the *point* of *tangency* of the latter branch, the pickets indicating the centre line of the straights.

It is required to supersede this compound by one provided with spirals that closely approximates the present location, and at the same time affords the same length of route.

Give the ruling points for the necessary improvement of the track.

Procedure.—1. Set up the theodolite over the *point-curvature a* and the *point-tangency a′*; and, back-sighting on the *point-compound-curvature c*, measure the deflection angles to the tangent, these being respectively half the central angles α and β as subtended by the D_o and d_o branches. But in spiralling old track, it is often impossible to verify satisfactorily the angles α and β, and, in consequence, such are to be taken at the magnitudes determined in the original location.

Hence, for purposes of calculation, let the angles α and β have their recorded values: 30° 12′ for the 3° branch, and 24° 18′ for the 3° 30′ branch.

$$R_o = 1910{\cdot}1 \text{ feet.}$$
$$r_o = 1637{\cdot}3 \text{ feet.}$$

2. Since the requirement of equal routes precludes the assuming of h, r_a, or R_a, proceed on this basis, adopting tentatively the time approach shift of 2·667 feet for the d_a curve, the speed on this sharper branch obtaining throughout the composite curve. Thus,

determine the amended radius r_a and the compound displacement h from the relations $h \cos \beta = (r_o - r_a)$ vers $\beta - s = \frac{1}{2}s$. (See Note ii., p. 93.)

Thus: $s = \frac{2}{3}(r_o - r_a)$ vers β.

$2{\cdot}667 = \frac{2}{3}(1637{\cdot}3 - r_a)$ vers 24° 18′.

Whence $(1637{\cdot}3 - r_a) = 45{\cdot}1$; and $r_a = 1592{\cdot}2$ feet.

Nearest to the foregoing value of r_a is 1591·8 feet to a degree of 3° 36′, and, this latter adopted, the shift s will be increased by ·02 foot to 2·687 feet, the length of transition l becoming accordingly $\sqrt{24 \times 1591{\cdot}8 \times 2{\cdot}687}$, or 320·4 feet.

3. Since the compound displacement h must be the same for the D_a curve, no central transition being necessary, determine the shift S of this branch from the equation $s \cos a = S \cos \beta$, and thence ascertain the amended radius R_a from the relation $S = \frac{2}{3}(R_o - R_a)$ vers a. Some modification will be necessary if the shift S leads to an incongruous length L of transition. As a rule, the disparity will accrue from excess length, and so will involve sliding the sharper branch along its tangent. (See p. 43.)

Now $$S = s \frac{\cos a}{\cos \beta}$$

$$= 2{\cdot}687 \frac{\cos 30° 12′}{\cos 24° 18′}$$

$$= 2{\cdot}687 \frac{{\cdot}8643}{{\cdot}9114} = 2{\cdot}548 \text{ feet.}$$

And $2{\cdot}548 = \frac{2}{3}(1910{\cdot}1 - R_a)$ vers 30° 12′.

Whence $(1910{\cdot}1 - R_a) = 28{\cdot}2$; and $R_a = 1881{\cdot}9$ feet.

Nearest to R_a is 1878 feet to degree of 3° 3′; but R_a as calculated must be retained, no change in the shift being

admissible; and, although the error in the deflection δ_1 per 100 feet would here be inappreciable in work of this nature, it may be advisable to calculate the precise value of δ_1, more particularly for ascertaining the length of the circular branch. Thus $D_a = 2\delta_1 = 5730 \div 1882 = 3° 2\frac{2}{3}'$. The length of transition L will be $\sqrt{24 \times 1881{\cdot}9 \times 2{\cdot}548}$, or 339·2 feet. Incidentally the normal lengths l and L would be 316·2 feet and 268·7 feet for the speed of the sharper branch; and the excess over the latter length is reasonably admissible with the conditions involved.

4. Calculate the required elements of the spirals and curves—viz., (*a*) the Tangent excesses *a*P and *a*′P′; (*b*) the Total deflections Ω and ω and the long chords PQ and P′Q′; (*c*) the Whole tangential angles Θ and θ; (*d*) the deflections δ_1, δ_2 per 100 feet; and (*e*) the Lengths L_c and l_c of the circular portions. Also compute the difference in length between the original and amended routes.

(*a*) Tangent excess $aP = \frac{1}{2}L\left(1 - \frac{1}{5}\frac{S}{R_a}\right) = \frac{1}{2}L = 169{\cdot}6$ feet.

Tangent excess $a'P' = \frac{1}{2}l\left(1 - \frac{1}{5}\frac{s}{r_a}\right) = \frac{1}{2}l = 160{\cdot}2$ feet.

(*b*) Total deflection $\Omega = 573\,\frac{L}{R_a} = 573\,\frac{339{\cdot}2}{1881{\cdot}9} = 1° 43{\cdot}27'$;

whence $\Phi = 5° 9{\cdot}81'$.

Total deflection $\omega = 573\,\frac{l}{r_a} = 573\,\frac{320{\cdot}4}{1591{\cdot}8} = 1° 55{\cdot}33'$;

whence $\phi = 5° 45{\cdot}99'$.

Long chord $PQ = L\left(1 - \frac{3}{5}\frac{S}{R_a}\right)\sec\Omega$

$= 339{\cdot}2\left(1 - \frac{3}{5}\frac{2{\cdot}548}{1881{\cdot}9}\right)\sec 1° 43{\cdot}3'$

$= 339{\cdot}0$ feet.

Long chord $P'Q' = l\left(1 - \frac{3}{5}\frac{s}{r_a}\right) \sec \omega$

$= 320{\cdot}4\left(1 - \frac{3}{5}\frac{2{\cdot}678}{1591{\cdot}8}\right) \sec 1^\circ\ 55{\cdot}3'$

$= 320{\cdot}1$ feet.

(*c*) Whole deflection Θ

$= \frac{1}{2}(\alpha - \Phi) = \frac{1}{2}(30^\circ\ 12' - 5^\circ\ 9{\cdot}8') = 12^\circ\ 31{\cdot}1'.$

Whole deflection θ

$= \frac{1}{2}(\beta - \phi) = \frac{1}{2}(24^\circ\ 18' - 5^\circ\ 46') = 9^\circ\ 16'.$

(*d*) Deflection angles per 100 feet chord:

$\delta_1 = 1^\circ\ 31\frac{1}{3}'$; $\delta_2 = 1^\circ\ 48'$.

(*e*) Length of circular portion $QC = \Theta \div \delta_1 = 822{\cdot}3$ feet.

Length of circular portion $CQ' = \theta \div \delta_2 = 514{\cdot}8$ feet.

Length of Original Half-Route P*ac*

$= \frac{1}{2}L\left(1 - \frac{1}{5}\frac{S}{R_a}\right) - (R_o - R_a)\tan\alpha + S\tan\alpha + R_o\alpha$

$= 169{\cdot}6 - 28{\cdot}2 \times {\cdot}5820 + 2{\cdot}548 \times {\cdot}5820 + 1910{\cdot}1 \times {\cdot}52705$

$= 1161{\cdot}4$ feet.

Length of Amended Half-Route PQC

$= R_a(\alpha + \Phi) = 1881{\cdot}9({\cdot}52705 + {\cdot}09011) = 1161{\cdot}4$ feet.

Length of Original Half-Route *ca*′P′

$= \frac{1}{2}l\left(1 - \frac{1}{5}\frac{s}{r_a}\right) - (r_o - r_a)\tan\beta + s\tan\beta + r_a\beta$

$= 160{\cdot}2 - 45{\cdot}5 \times {\cdot}45152 + 2{\cdot}687 \times {\cdot}45152 + 1637{\cdot}3 \times {\cdot}42408$

$= 835{\cdot}2$ feet.

Length of Amended Half-Route CQ′P′

$= r_a(\beta + \phi) = 1591{\cdot}8({\cdot}42408 + {\cdot}10064) = 835{\cdot}3$ feet.

5. Measure from the tangent points *a* and *a*′ the respective tangent excesses *a*P and *a*′P′, and establish with pegs the points P and P′.

6. Set up over the *point-spiral* P and, backsighting along the tangent, set out the total deflection Ω. Measure along the direction thus given the long chord PQ, and establish the *point-spiral-curve* Q.

6 (*a*). If possible, set out from the tangent at Q the whole deflection $\frac{1}{2}(a=\Phi)$, and line out this direction near the original *point-compound-curvature c*. Otherwise, the amended point-compound-curvature C will have to be verified in the running of the amended branches.

7. Set up over the *point-spiral-tangent* P′, and, backsighting along the tangent, set out the total deflection ω. Measure along the direction thus given the long chord P′Q′, and establish the *point-curve-spiral* Q′.

7 (*a*). If possible, set out from the tangent at Q′ the whole deflection $\frac{1}{2}(\beta-\phi)$, and determine the intersection C with the direction lined out in 6 (*a*). Measure as *h*, the displacement *c*C, which should be equal to $\frac{s}{2\cos\beta}$ or $\frac{S}{2\cos a}$.

OBSERVATIONS.

TABLE I.—SPIRAL ELEMENTS

Radius (R).	Length $(L) = \sqrt{R}$.	Terminal Deflection Angle $(\Omega) = \frac{573}{\sqrt{R}}$.	Terminal Abscissa X $L\left(1 - \frac{3}{5}\frac{s}{R}\right)$.	Long Chord X sec Ω.
Chains.	Chains.	Minutes.	Chains.	Chains.
20	4·472	128·13	4·466	4·469
21	4·583	125·03	4·578	4·581
22	4·690	122·17	4·685	4·688
23	4·796	119·47	4·791	4·794
24	4·899	116·96	4·894	4·897
25	5·000	114·59	4·995	4·998
26	5·099	112·37	5·094	5·097
27	5·196	110·27	5·191	5·194
28	5·292	108·27	5·287	5·290
29	5·385	106·40	5·380	5·383
30	5·477	104·61	5·472	5·475
31	5·568	102·90	5·563	5·566
32	5·657	101·29	5·653	5·656
33	5·745	99·73	5·741	5·744
34	5·831	98·26	5·827	5·829
35	5·916	96·85	5·912	5·914
36	6·000	95·50	5·996	5·998
37	6·083	94·20	6·079	6·081
38	6·164	92·95	6·160	6·162
39	6·245	91·75	6·241	6·243
40	6·325	90·60	6·321	6·323
41	6·403	89·49	6·399	6·401
42	6·481	88·42	6·477	6·479
43	6·557	87·24	6·553	6·555
44	6·633	86·39	6·629	6·631
45	6·708	85·42	6·704	6·706
46	6·782	84·49	6·778	6·780
47	6·856	83·58	6·852	6·854
48	6·928	82·70	6·924	6·926
49	7·000	81·86	6·996	6·998
50	7·071	81·03	7·067	7·069

AND DATA: RADIUS SYSTEM.

Spiral Extension $\frac{L}{2}\left(1-\frac{1}{5}\frac{s}{R}\right)$.	Deflection per Chord^2 $(k)=\frac{573}{R\frac{1}{2}}$.	Chord Length $(c)=\frac{1}{24}R^{\frac{1}{2}}$.	Number of Chords (n).
Chains.	Minutes.	Chains.	
2·235	6·407	·395	11·312
2·291	5·954	·410	11·182
2·234	5·553	·424	11·053
2·397	5·194	·439	10·930
2·449	4·873	·453	10·815
2·499	4·584	·467	10·704
2·549	4·332	·481	10·600
2·597	4·084	·495	10·501
2·645	3·867	·508	10·405
2·692	3·669	·522	10·315
2·738	3·487	·535	10·228
2·784	3·319	·549	10·144
2·828	3·165	·562	10·064
2·872	3·022	·575	9·986
2·915	2·978	·588	9·913
2·957	2·767	·601	9·840
3·000	2·653	·614	9·772
3·041	2·540	·627	9·705
3·082	2·451	·639	9·640
3·122	2·352	·652	9·579
3·162	2·265	·664	9·517
3·201	2·182	·677	9·458
3·240	2·105	·689	9·404
3·278	2·032	·701	9·340
3·316	1·963	·714	9·292
3·354	1·898	·726	9·243
3·391	1·836	·734	9·191
3·428	1·779	·750	9·141
3 464	1·723	·762	9·095
3·500	1·670	·774	9·046
3·535	1·621	·785	9·002

TABLE I.—SPIRAL ELEMENTS

Radius (R).	Length $(L)=\frac{414}{R}$.	Terminal Deflection Angle $(\Omega)=\left(\frac{487}{R}\right)^2$.	Terminal Abscissa X $L\left(1-\frac{3}{5}\frac{s}{R}\right)$.	Long Chord X sec Ω.
Chains.	Chains.	Minutes.	Chains.	Chains.
51	7·141	80·24	7·137	7·139
52	7·211	79·46	7·208	7·210
53	7·280	78·71	7·277	7·279
54	7·348	77·98	7·345	7·347
55	7·416	77·26	7·413	7·415
56	7·393	75·64	7·390	7·392
57	7·261	73·01	7·258	7·260
58	7·138	70·51	7·135	7·137
59	7·017	68·14	7·014	7·015
60	6·900	65·89	6·898	6·899
61	6·786	63·75	6·784	6·785
62	6·677	61·71	6·675	6·676
63	6·571	59·77	6·569	6·570
64	6·468	57·91	6·466	6·467
65	6·369	56·15	6·367	6·368
66	6·277	54·46	6·276	6·277
67	6·181	52·84	6·180	6·181
68	6·088	51·30	6·087	6·088
69	5·990	49·82	5·989	5·990
70	5·914	48·41	5·913	5·914
71	5·830	47·05	5·829	5·830
72	5·749	45·76	5·748	5·749
73	5·671	44·51	5·670	5·671
74	5·594	43·32	5·593	5·593
75	5·519	42·12	5·518	5·518
76	5·446	41·07	5·445	5·445
77	5·376	40·01	5·375	5·375
78	5·308	38·99	5·307	5·307
79	5·240	38·01	5·239	5·239
80	5·175	37·06	5·174	5·174

AND DATA: RADIUS SYSTEM—*Continued.*

Spiral Extension $\frac{L}{2}\left(1-\frac{1}{5}\frac{s}{R}\right)$.	Deflection per Chain² $(k)=\frac{573}{LR}$.	Chord Length $(c)=\frac{1}{24}\sqrt{RL}$.	Number of Chords (n).
Chains.	Minutes.	Chains.	
3·570	1·573	·797	8·958
3·605	1·528	·809	8·913
3·640	1·485	·821	8·872
3·674	1·444	·832	8·818
3·708	1·405	·844	8·788
3·696	1·384	·850	8·967
3·630	1·384	·850	8·545
3·569	1·384	·850	8·397
3·508	1·384	·850	8·255
3·450	1·384	·850	8·117
3·393	1·384	·850	7·984
3·338	1·384	·850	7·856
3·285	1·384	·850	7·731
3·234	1·384	·850	7·610
3·184	1·384	·850	7·492
3·138	1·384	·850	7·379
3·090	1·384	·850	7·269
3·044	1·384	·850	7·162
2·995	1·384	·850	7·059
2·957	1·384	·850	6·958
2·915	1·384	·850	6·860
2·874	1·384	·850	6·764
2·835	1·384	·850	6·672
2·797	1·384	·850	6·582
2·759	1·384	·850	6·494
2·723	1·384	·850	6·408
2·688	1·384	·850	6·325
2·654	1·384	·850	6·244
2·620	1·384	·850	6·165
2·587	1·384	·850	6·088

TABLE II.—SPIRAL ELEMENTS

Degree (D°).	Length $(L) = 1000 \frac{D.}{3}$	Terminal Deflection Angle $(\Omega) = \frac{1}{10} LD.$	Terminal Abscissa X $L\left(1 - \frac{3}{5}\frac{s}{R}\right)$.	Long Chord X sec Ω.
	Feet.	Minutes.	Feet.	Feet.
1° 00′	333·3	33·33	333·3	333·3
1° 05′	361·1	39·12	361·1	361·1
1° 10′	388·9	45·37	388·8	388·9
1° 15′	416·7	52·08	416·6	416·7
1° 20′	444·4	59·26	444·3	444·4
1° 25′	472·2	66·90	472·1	472·2
1° 30′	489·8	73·48	489·6	489·7
1° 35′	476·6	75·12	476·4	476·5
1° 40′	464·9	77·45	464·7	464·8
1° 45′	454·5	79·25	454·3	454·4
1° 50′	443·2	81·23	443·0	443·1
1° 55′	434·7	82·75	434·5	434·6
2° 00′	424·3	84·86	424·1	424·2
2° 05′	416·6	86·40	416·4	416·5
2° 10′	407·6	88·32	407·4	407·5
2° 15′	400·0	90·00	399·8	399·9
2° 20′	392·8	91·64	392·5	392·6
2° 25′	387·0	93·10	386·7	386·8
2° 30′	379·5	94·87	379·2	379·3
2° 35′	375·0	96·48	374·7	374·8
2° 40′	367·5	97·97	367·2	367·3
2° 45′	361·4	99·50	361·1	361·3
2° 50′	356·5	101·01	356·2	356·4
2° 55′	350·9	102·25	356·8	357·0
3° 00′	346·4	103·92	346·1	346·3
3° 05′	342·8	105·30	342·5	342·7
3° 10′	337·2	106·76	336·9	337·1
3° 15′	333·3	108·17	333·0	333·2
3° 20′	328·6	109·55	328·3	328·5
3° 25′	326·0	110·90	325·7	325·9

AND DATA: DEGREE SYSTEM.

Spiral Extension $\frac{L}{2}\left(1-\frac{1}{5}\frac{s}{R}\right)$.	Deflection per 100 feet^2 $(k')=\frac{10D}{L}$.	Chord Length $(c')=\sqrt{\frac{10L}{D}}$	Number of Chords (n).	Radius (R').
Feet.	Minutes.	Feet.		Feet.
166·6	3·000	57·74	5·840	5729·6
180·5	3·000	57·74	6·255	5288·9
194·4	3·000	57·74	6·736	4911·2
208·3	3·000	57·74	7·217	4583·8
222·2	3·000	57·74	7·698	4297·3
236·1	3·000	57·74	8·179	4044·5
245·0	3·062	57·14	8·572	3819·8
239·5	3·320	54·86	8·667	3618·8
232·4	3·586	52·81	8·801	3437·9
227·2	3·866	50·98	8·902	3274·2
221·6	4·126	49·17	9·013	3125·4
217·3	4·422	47·32	9·096	2989·5
212·1	4·714	46·06	9·212	2864·9
208·3	5·012	44·72	9·295	2750·4
203·8	5·315	43·37	9·398	2644·6
200·0	5·615	42·15	9·487	2546·6
196·4	5·941	41·03	9·573	2455·7
193·5	6·261	40·02	9·660	2371·0
189·7	6·580	38·96	9·740	2293·0
187·5	6·935	38·10	9·822	2218·1
183·7	7·258	37·12	9·898	2148·8
180·7	7·601	36·25	9·915	2083·7
178·2	7·959	35·47	10·049	2022·4
175·5	8·331	34·65	10·114	1964·6
173·2	8·660	33·98	10·193	1910·1
171·4	9·024	33·37	10·261	1858·5
168·6	9·392	32·63	10·338	1809·6
166·6	9·765	32·02	10·397	1763·2
164·3	10·143	31·40	10·458	1719·1
162·9	10·526	30·89	10·531	1667·3

TABLE II.—SPIRAL ELEMENTS

Degree (D°).	Length $(L)=\frac{600}{\sqrt{D}}$.	Terminal Deflection Angle $(\Omega)=60\sqrt{D}$.	Terminal Abscissa X $L\left(1-\frac{3}{5}\frac{s}{R}\right)$.	Long Chord X sec Ω.
	Feet.	Minutes.	Feet.	Feet.
3° 30′	320·7	112·25	320·4	320·6
3° 35′	317·4	113·58	317·1	317·3
3° 40′	313·4	114·88	313·1	313·3
3° 45′	310·8	116·20	310·5	310·7
3° 50′	306·5	117·31	306·2	306·4
3° 55′	303·2	118·80	302·9	303·1
4° 00′	300·0	120·00	299·7	299·9
4° 05′	297·3	121·15	297·0	297·2
4° 10′	293·9	122·48	293·6	293·8
4° 15′	291·5	123·69	291·2	291·4
4° 20′	288·2	124·91	287·9	288·1
4° 25′	285·7	126·01	285·4	285·6
4° 30′	282·8	127·28	282·5	282·7
4° 35′	280·8	128·40	280·5	280·7
4° 40′	279·5	128·78	279·2	279·4
4° 45′	275·2	130·76	274·8	275·0
4° 50′	272·9	131·90	272·5	272·7
4° 55′	271·0	133·00	270·6	270·8
5° 00′	268·3	134·16	267·9	268·1
5° 05′	266·3	135·25	265·9	266·1
5° 10′	263·8	136·40	263·4	263·6
5° 15′	262·0	137·52	261·6	261·8
5° 20′	259·8	138·56	259·4	259·6
5° 25′	257·8	139·64	257·4	257·6
5° 30′	255·8	140·70	255·4	255·6
5° 35′	254·2	141·81	253·8	254·0
5° 40′	252·0	142·82	251·6	251·8
5° 45′	250·2	143·91	249·8	250·0
5° 50′	248·4	144·86	248·0	248·2
5° 55′	246·7	145·92	246·3	246·5

AND DATA: DEGREE SYSTEM—*Continued.*

Spiral Extension $\frac{L}{2}\left(1-\frac{1}{5}\frac{s}{R}\right)$.	Deflection per 100 feet^2 $(k')=\frac{5}{8}D^2$.	Chord Length $(c')=\sqrt{\frac{10L.}{D}}$	Number of Chords (n).	Radius (R).
Feet.	Minutes.	Feet.		Feet.
160·3	10·859	30·27	10·593	1637·3
158·6	11·316	29·76	10·658	1599·2
156·6	11·702	29·24	10·718	1562·9
155·3	12·103	28·82	10·778	1528·2
153·2	12·509	28·28	10·831	1495·0
151·5	12·963	27·79	10·899	1463·2
149·9	13·333	27·39	10·954	1432·7
148·6	13·752	26·90	11·007	1403·5
146·9	14·178	26·56	11·067	1375·4
145·7	14·603	26·19	11·122	1348·4
144·0	15·032	25·75	11·176	1322·5
142·8	15·470	25·44	11·225	1297·6
141·3	15·909	25·07	11·293	1273·6
140·3	16·343	24·75	11·331	1250·4
139·7	16·803	24·47	11·349	1228·1
137·5	17·255	24·07	11·437	1206·6
136·4	17·710	23·76	11·485	1185·8
135·4	18·170	23·48	11·533	1165·7
134·1	18·634	23·16	11·583	1146·3
133·1	19·101	22·89	11·629	1127·5
131·8	19·573	22·60	11·679	1109·3
130·9	20·048	22·34	11·727	1091·7
129·8	20·527	22·07	11·771	1074·7
128·8	21·010	21·82	11·817	1058·2
127·8	21·498	21·57	11·862	1042·1
127·0	21·998	21·33	11·908	1026·6
125·9	22·483	21·09	11·951	1011·5
125·1	22·980	20·87	11·996	996·9
124·1	23·483	20·64	12·036	982·6
123·3	23·986	20·43	12·080	968·8

INDEX

Abscissa, terminal, 6, 32
Adjustment, gradient of, 6, 11, 12, 61
Alignment, 35-50
Amended length of route, 47, 92, 98
Amended radius or degree, 36, 44, 47, 77, 81, 87, 91, 95
Angle, intersection, 5, 36
  total deflection, 6, 15, 21, 32, 100
  total spiral, 5, 15, 31
  whole tangential, 54, 58, 93
Angles, central, 40, 48, 81
  deflection, 15, 17, 18, 21, 24, 32, 100
  tangential, 53, 57, 64
Apex of curve, 4, 36, 40, 44, 77, 93
Appendix, 100

Backsighting, 21, 53, 58, 70
Bogey, 1

Cant, 1, 2, 12
Cartesian co-ordinates, 15
Cases of alignment, 35
Central angles, 40, 58, 81
Central shifts, 39, 40, 74
Central transition, 38, 76
Centrifugal force, 1
Chainage, 20, 51, 56, 64, 65, 72
"Chaining through," 19, 51
Chord, 17-25, 32
  long, 6, 25, 32
  whole, 54, 70, 77, 80
Chordage of station points, 25-28, 59, 72, 84
Chord lengths, 17, 18, 23, 56, 69
Chord offsets, 27, 59, 64
Clothoid, 13
Compound curves, 37-43, 45, 48, 66, 73, 95
Construction, 17-30
  by chord-deflections, 18, 19, 23, 32, 51, 56, 66, 73, 77, 81
  by offsets, 28, 61, 76
Cubic parabola, 16, 17, 28, 61, 76

Data of spirals, 32, 52, 56, 62, 68, 75, 77, 83, 89, 92, 97, 100
Data of time approaches, 11, 12, 32, 100
Deflection angle, total, 6, 15, 21, 32, 100, 104
Deflections, 15, 17, 18, 21, 24, 32, 100
Degree of curvature: degree system, 2, 3, 12, 18, 19, 22, 32, 104
Development, 13-34
Displacement, 36, 40, 91, 93, 99

Elements of spirals, 5, 30, 100
Elevation of outer rail, 1, 2, 6, 7, 11, 12
End shifts, 40, 73
Equation of clothoid, 13
Equation of equal routes, 48, 94, 96
Excess, tangent, 5, 31, 37, 40, 45
Extension, spiral, 5, 31, 37, 40, 100, 104

Field exercises, 51-99
Froude's curve of adjustment, 28, 43
Fundamental case of spiralling, 4, 39

Gap in shifting, 41, 74
Gauge, standard, 2
General case of alignment, 36
General construction, 36, 49
Gradient of adjustment, or approach, 6, 11, 12, 61

Inaccessible intersection, 67
Increment, shift, 5, 11, 40
Initial deflections, 18, 19, 32, 33
Interpolating stations, 25-28, 59, 72, 84
Intersection angle, 5, 36
Intrinsic equation of clothoid, 13
Introduction, 1-12

Length of circular curves, 16, 47, 52, 57, 92, 98
Length of transition, 6-12, 32, 38, 74, 100
Limiting speeds, 9, 10
Long chord, 6, 25, 32, 100

Main tangent, 4, 36
Maximum speed, 2, 11, 12, 32
Maximum superelevation, 2, 3
Minimum shift, 74
Minute-reading theodolite, 33
Modes of construction, 17-30
  with chain, 28
  with theodolite, 17

Normal deflections, 21, 24
Normal shift, 11, 12, 32
Notation, 33

Objects of transition curve, 4
Offsets from the chord, 27, 59
  from the tangent, 28, 62
Offsetting a transition curve, 28, 62, 76
Ordinate, terminal, 6, 16, 32
Original length of route, 47, 92, 98
Original radius or degree, 36, 44, 47, 77, 81, 87, 91, 95

Points, spiral, 17, 21, 25
  station, 25-28, 59, 72, 84
Polychord spirals, 17
Problems, field, 51-99
Problems in alignment, 35

Radius of curvature: radius system, 2, 9, 11, 20, 100
  amended radius, 36, 44, 47
  original radius, 36, 44, 47
Redundant portion of curve, 30, 64, 76
Reverse curves, 43, 46, 49, 87
Routes, amended and original, 47, 92, 98
  equation of equal, 48, 94, 96
Rules, for length of transition, 6, 32, 38
  for maximum speed, 3, 11, 12
  for superelevation, 1, 2, 11, 12
  Shortt's rule, 8, 11, 39

Shift, 5, 11, 12, 16, 30, 40, 44, 49, 63, 74
Shift centre, 74, 76
  increment, 5, 11, 31, 40
  tangent, 4, 63
"Shifting forward," 21
Shortt's rule, 8, 11, 39
Simple curves, 39, 44, 47, 51, 56, 61, 77, 91
Speed on curves, 3, 11, 12
"Spiralling in," 19, 23, 56, 66
"Spiralling through," 19, 51, 81
Spiral angle, 5, 15, 31, 100, 104
Spiral elements, 5, 30, 100, 104
Spiral extension, 5, 40, 31, 37, 100, 104
Spiral notation, 33
Spiral reverse, 43, 49, 87
Standard gauge, 2
Superelevation, 1, 2, 6, 7, 11, 12

Tables, 11, 12, 24, 32, 100, 104
Tangent, 3, 4, 35, 43
at any point, 21, 24, 25
at origin, 13, 21
shift tangent, 4, 62
Tangential angles, 53, 57, 64
whole tangential angle, 54, 58, 93
Tangent distances, 6, 31, 51, 56, 62, 66
Tangent excess, 5, 31, 37, 40, 45
Tangent offsets, 28, 62
Terminal abscissa, 6, 32
Terminal deflection angle, 6, 15, 21, 32, 100, 104
Terminal ordinate, 6, 16, 32
Time approach, 7, 11, 12, 32
Total spiral angle, 5, 15, 31
Total tangent lengths, 6, 31, 51, 57, 69
Transition, 4

Uniform approach, 6, 61

Velocity on curves, 1, 3, 11, 12, 32
Vertex of curve, 4, 36, 40, 44, 77, 93

Whole cord, 54, 70, 77, 80
Widening of gauge, 4

PRINTED IN GREAT BRITAIN BY
BILLING AND SONS, LIMITED,
GUILDFORD AND ESHER

D. VAN NOSTRAND COMPANY

are prepared to supply, either from their complete stock or at short notice,

Any Technical or Scientific Book

In addition to publishing a very large and varied number of SCIENTIFIC AND ENGINEERING BOOKS, D. Van Nostrand Company have on hand the largest assortment in the United States of such books issued by American and foreign publishers.

All inquiries are cheerfully and carefully answered and complete catalogs sent free on request.

8 WARREN STREET - - - NEW YORK